GOVERNMENT OF TAMIL NADU

HIGHER SECONDARY SECOND YEAR

MATHEMATICS

VOLUME - I

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Department of School Education

Untouchability is Inhuman and a Crime

Government of Tamil Nadu

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NOT FOR SALE

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HOW TO USE THE BOOK ?

Scope of Mathematics

- Awareness on the scope of higher educational opportunities; courses, institutions and required competitive examinations.
- Possible financial assistance to help students climb academic ladder.

Learning Objectives

- Overview of the unit
- Give clarity on the intended learning outcomes of the unit.

- Visual representation of concepts with illustrations
- Videos, animations, and tutorials.

ICT

- To increase the span of attention of concepts
- To visualize the concepts for strengthening and understanding
- To link concepts related to one unit with other units.
- To utilize the digital skills in classroom learning and providing students experimental learning.

Summary

- Recapitulation of the salient points of each chapter for recalling the concepts learnt.

Evaluation

- Assessing student's understanding of concepts and get them acquainted with solving exercise problems.

Books for Reference

- List of relevant books for further reading.

Scope for Higher Order Thinking

- To motivate students aspiring to take up competitive examinations such as JEE, KVPY, Math olympiad, etc., the concepts and questions based on Higher Order Thinking are incorporated in the content of this book.

Glossary

- Frequently used Mathematical terms have been given with their Tamil equivalents.

Mathematics Learning

The correct way to learn is to understand the concepts throughly. Each chapter opens with an Introduction, Learning Objectives, Various Definitions, Theorems, Results and Illustrations. These in turn are followed by solved examples and exercise problems which have been classified in to various types for quick and effective revision. One can develop the skill of solving mathematical problems only by doing them. So the teacher's role is to teach the basic concepts and problems related to it and to scaffold students to try the other problems on their own. Since the second year of Higher Secondary is considered to be the foundation for learning higher mathematics, the students must be given more attention to each and every concept mentioned in this book.

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MATHEMATICS

E-book **Assessment**

Scope for students after completing Higher Secondary

EXAMS

Joint Entrance Examination (JEE) Main	
Purpose	For Admission in B. E./B. Tech., B. Arch., B. Planning
Eligibility	Class 12 pass (PCM)
Application Mode	Online
Source	http://jeemain.nic.in

JEE Advanced	
Purpose	Admission in UG programmes in IITs and ISM Dhanbad
Eligibility	Class 12 Pass (PCM)
Application Mode	Online
Source	http://jeeadv.iitd.ac.in/

Indian Maritime University Common Entrance Test	
Purpose	Admission in Diploma in Nautical Science (DNS) leading to BSc. (Nautical Science)
Eligibility	Class 12 (PCM)
Application Mode	By post
Source	www.imu.edu.in/index.php

Indian Navy B.Tech Entry Scheme	
Purpose	Admission in Indian Navy B.Tech course
Eligibility	Class 12 passed
Application Mode	Online
Source	www.nausena-bharti.nic.in/index.php

EXAMS

Indian Navy Sailors Recruitment	
Purpose	Admission in 24 weeks Basic training at INS Chilka followed by Professional training
Eligibility	Class 12 (Maths and Physics/Chem./Bio/Computer Sc.)
Application Mode	Online, By Post
Source	www.nausena-bharti.nic.in/index.php

Defence Indian Army Technical Entry Scheme (TES)	
Purpose	Technical Entry to Army
Eligibility	Class 12 PCM
Application Mode	Online
Source	www.joinindianarmy.nic.in/

National Defence Academy and Naval Academy Examination (I)	
Purpose	Admission to Army and Air Force wings of NDA and 4 years B. Tech course for the Indian Naval Academy Course (INAC)
Eligibility	Class 12 Passed
Application Mode	Online
Source	www.upsc.gov.in/

Scope for students after completing Higher Secondary

EXAMS

- If you have aspiration to become a scientist/teacher, you can do degree course in mathematics in any college of your choice. Doing B.Sc., mathematics with your knowledge which acquired in XII standard mathematics will definitely elevate you to a better career.
- There are some institutions such as IIT's, IISc., ISI's and Anna University which admit students at XII Standard level for their Integrated Courses leading to the award of M.Sc., degree in mathematics/engineering/Statistics/Computer Science .

  In Mathematics, the following degrees programmes are offered:

- 3-year BSc	- 4-year Integrated B.Sc-B.Ed.
- 3-year B. Math	- 4-year BS
- 4-year B. Tech	- 5-year Integrated M.Sc /MS

BITSAT

Purpose- For admission to Integrated First Degree Programmes in BITS Pilani, Goa & Hyderabad campuses.

National Aptitude Test in Architecture (NATA)

Purpose- Admission to B.Arch. program

Kishore Vaigyanik Protsahan Yojana (KVPY)

Purpose- Fellowship and admission to IISc Banglore in 4 year BS Degree

Indian Statistical Institute (ISI) Admission

Purpose-Admission in B Stat (Hons), B Math (Hons)

Chennai Mathematical Institute

Purpose- Admission in BSc (Hons) in Mathematics and Computer Science, BSc (Hons) in Mathematics and Physics

Job Opportunities

- You have acquired sufficient analytical still to appear for competitive examinations conducted by various organizations such as TNPSC, UPSC, BSRB, RSB.
- You are placed with better opportunity to get recruitment directly into some of the top Multi National Companies.
- You can plan to take up engineering degree course to acquire BE/B.TECH in any of the leading Technical Institutions/Universities/Engineering colleges. Institutions such as IIT's , IISc, NIT's conduct All India Level Entrance Examinations to admit students.

• Indian Forest Services	• Scientist Jobs in ISRO, DRDO, CSIR labs
• Union Public Service Commission	• Staff Selection Commission
• Indian Defense Services	• Public Sector Bank
• Tax Assistant	• Statistical Investigator
• Combined Graduate Level Exam	• Tamil Nadu Public Service Commission
• Teaching Profession	

Scholarship and Research Opportunities

FINANCIAL ASSISTANCE

- Scholarship for Graduate and Post – Graduate courses
- NTSE at the end of X (from class XI to Ph.D)
- International Olympiad: for getting stipend for Higher Education in Science and Mathematics
- DST – INSPIRE Scholarships (for UG and PG)
- DST – INSPIRE Fellowships (for Ph.D)
- UGC National Fellowships (for Ph.D)
- Indira Gandhi Fellowship for Single Girl Child (for UG and PG)
- Moulana Azad Fellowship for minorities (for Ph.D)
- In addition various fellowships for SC / ST / PWD / OBC etc are available. (Visit website of University Grants Commission (UGC) and Department of Science and Technology (DST))
- University Fellowships
- Tamil Nadu Collegiate Education fellowship.

RESEARCH INSTITUTIONS

NAME OF THE INSTITUTION	WEBSITE
Indian Institute of Science (IISc) Bangalore	www.iisc.ac.in
Chennai Mathematical Institute (CMI) Chennaie	www.cmi.ac.in
Tata Institue of Fundamental Research (TIFR) Mumbai	www.tifr.res.in
Indian Institute of Space Science and Technology (IIST) Trivandrum	www.iist.ac.in
National Institute of Science Education and Research (NIS-ER)	www.niser.ac.in
Birla Institute of Technology and Science, Pilani	www.bits-pilani.ac.in
Indian Institute of Science Education and Research	www.iiseradmission.in
Anna University	https://www.annauniv.edu/
Indian Institute of Technology in various places (IIT's)	www.iitm.ac.in
National Institute of Technology (NITs)	www.nitt.edu
Central Universities	www.cucet.ac.in
State Universities	www.ugc.ac.in
Tamil Nadu agricultural University (tnau.ac.in)	tnau.ac.in
International Institute of Information Technology	www.iiit.ac.in
The Institute of Mathematical Sciences (IMSC) Chennai.	www.imsc.res.in
Hyderabad Central university, Hyderabad.	www.uohyd.ac.in
Delhi University, Delhi	www.du.ac.in
Mumbai University, Mumbai	www.mu.ac.in
Savithiri Bai Phule Pune University, Pune	www.unipune.ac.in

Chapter 1

Applications of Matrices and Determinants

"The greatest mathematicians, as Archimedes, Newton, and Gauss, always united theory and applications in equal measure."

-Felix Klein

1.1 Introduction

Carl Friedrich Gauss
(1777-1855)
German mathematician and physicist

Matrices are very important and indispensable in handling system of linear equations which arise as mathematical models of real-world problems. Mathematicians Gauss, Jordan, Cayley, and Hamilton have developed the theory of matrices which has been used in investigating solutions of systems of linear equations.

In this chapter, we present some applications of matrices in solving system of linear equations. To be specific, we study four methods, namely (i) Matrix inversion method, (ii) Cramer's rule (iii) Gaussian elimination method, and (iv) Rank method. Before knowing these methods, we introduce the following: (i) Inverse of a non-singular square matrix, (ii) Rank of a matrix, (iii) Elementary row and column transformations, and (iv) Consistency of system of linear equations.

Learning Objectives

Upon completion of this chapter, students will be able to

- Demonstrate a few fundamental tools for solving systems of linear equations:
  - Adjoint of a square matrix
  - Inverse of a non-singular matrix
  - Elementary row and column operations
  - Row-echelon form
  - Rank of a matrix
- Use row operations to find the inverse of a non-singular matrix
- Illustrate the following techniques in solving system of linear equations by
  - Matrix inversion method
  - Cramer's rule
  - Gaussian elimination method
- Test the consistency of system of non-homogeneous linear equations
- Test for non-trivial solution of system of homogeneous linear equations

1.2 Inverse of a Non-Singular Square Matrix

We recall that a square matrix is called a **non-singular matrix** if its determinant is not equal to zero and a square matrix is called **singular** if its determinant is zero. We have already learnt about multiplication of a matrix by a scalar, addition of two matrices, and multiplication of two matrices. But a rule could not be formulated to perform division of a matrix by another matrix since a matrix is just an arrangement of numbers and has no numerical value. When we say that, a matrix A is of order n, we mean that A is a square matrix having n rows and n columns.

In the case of a real number $x \neq 0$, there exists a real number $y\left(=\frac{1}{x}\right)$, called the inverse (or reciprocal) of x such that $xy = yx = 1$. In the same line of thinking, when a matrix A is given, we search for a matrix B such that the products AB and BA can be found and $AB = BA = I$, where I is a unit matrix.

In this section, we define the inverse of a non-singular square matrix and prove that a non-singular square matrix has a unique inverse. We will also study some of the properties of inverse matrix. For all these activities, we need a matrix called the adjoint of a square matrix.

1.2.1 Adjoint of a Square Matrix

We recall the properties of the cofactors of the elements of a square matrix. Let A be a square matrix of by order n whose determinant is denoted $|A|$ or $\det(A)$. Let a_{ij} be the element sitting at the intersection of the i^{th} row and j^{th} column of A. Deleting the i^{th} row and j^{th} column of A, we obtain a sub-matrix of order $(n-1)$. The determinant of this sub-matrix is called **minor** of the element a_{ij}. It is denoted by M_{ij}. The product of M_{ij} and $(-1)^{i+j}$ is called **cofactor** of the element a_{ij}. It is denoted by A_{ij}. Thus **the cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.**

An important property connecting the elements of a square matrix and their cofactors is that the sum of the products of the entries (elements) of a row and the corresponding cofactors of the elements of the same row is equal to the determinant of the matrix; and the sum of the products of the entries (elements) of a row and the corresponding cofactors of the elements of any other row is equal to 0. That is,

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

where $|A|$ denotes the determinant of the square matrix A. Here $|A|$ is read as "determinant of A" and not as " modulus of A". Note that $|A|$ is just a real number and it can also be negative. For instance, we have $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 2(1-2) - 1(1-2) + 1(2-2) = -2 + 1 + 0 = -1.$

Definition 1.1

Let A be a square matrix of order n. Then the **matrix of cofactors** of A is defined as the matrix obtained by replacing each element a_{ij} of A with the corresponding cofactor A_{ij}. The **adjoint matrix** of A is defined as the transpose of the matrix of cofactors of A. It is denoted by $\operatorname{adj} A$.

Note

adj A is a square matrix of order n and adj $A = \left[A_{ij}\right]^T = \left[(-1)^{i+j} M_{ij}\right]^T$.

In particular, adj A of a square matrix of order 3 is given below:

$$\text{adj } A = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

Theorem 1.1

For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$.

Proof

For simplicity, we prove the theorem for $n = 3$ only.

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then, we get

$$\begin{aligned} &a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|, \quad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0, \quad a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0; \\ &a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = 0, \quad a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = |A|, \quad a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} = 0; \\ &a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} = 0, \quad a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = 0, \quad a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = |A|. \end{aligned}$$

By using the above equations, we get

$$A(\text{adj}A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3 \qquad \ldots (1)$$

$$(\text{adj}A)A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3, \qquad \ldots (2)$$

where I_3 is the identity matrix of order 3.

So, by equations (1) and (2), we get $A(\text{adj } A) = (\text{adj } A)A = |A| I_3$. ■

Note

If A is a singular matrix of order n, then $|A| = 0$ and so $A(\text{adj } A) = (\text{adj } A)A = O_n$, where O_n denotes zero matrix of order n.

Example 1.1

If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A| I_3$.

Solution

We find that $|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21-16) + 6(-18+8) + 2(24-14) = 40 - 60 + 20 = 0.$

By the definition of adjoint, we get

$$\text{adj } A = \begin{bmatrix} (21-16) & -(-18+8) & (24-14) \\ -(-18+8) & (24-4) & -(-32+12) \\ (24-14) & -(-32+12) & (56-36) \end{bmatrix}^T = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}.$$

So, we get

$$A(\text{adj } A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I_3 = |A| I_3,$$

Similarly, we get

$$(\text{adj } A)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 40-60+20 & -30+70-40 & 10-40+30 \\ 80-120+40 & -60+140-80 & 20-80+60 \\ 80-120+40 & -60+140-80 & 20-80+60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I_3 = |A| I_3.$$

Hence, $A(\text{adj } A) = (\text{adj } A)A = |A| I_3$.

1.2.2 Definition of inverse matrix of a square matrix

Now, we define the inverse of a square matrix.

Definition 1.2

Let A be a square matrix of order n. If there exists a square matrix B of order n such that $AB = BA = I_n$, then the matrix B is called an **inverse** of A.

Theorem 1.2

If a square matrix has an inverse, then it is unique.

Proof

Let A be a square matrix order n such that an inverse of A exists. If possible, let there be two inverses B and C of A. Then, by definition, we have $AB = BA = I_n$ and $AC = CA = I_n$.

Using these equations, we get

$$C = CI_n = C(AB) = (CA)B = I_nB = B.$$

Hence the uniqueness follows.

Notation The inverse of a matrix A is denoted by A^{-1}.

Note

$AA^{-1} = A^{-1}A = I_n$.

Theorem 1.3

Let A be square matrix of order n. Then, A^{-1} exists if and only if A is non-singular.

Proof

Suppose that A^{-1} exists. Then $AA^{-1} = A^{-1}A = I_n$.

By the product rule for determinants, we get

$\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1})\det(A) = \det(I_n) = 1$. So, $|A| = \det(A) \neq 0$.

Hence A is non-singular.

Conversely, suppose that A is non-singular.

Then $|A| \neq 0$. By Theorem 1.1, we get

$$A(\text{adj } A) = (\text{adj } A)A = |A| I_n.$$

So, dividing by $|A|$, we get $A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right)A = I_n$.

Thus, we are able to find a matrix $B = \frac{1}{|A|}\text{adj } A$ such that $AB = BA = I_n$.

Hence, the inverse of A exists and it is given by $\boldsymbol{A^{-1} = \frac{1}{|A|}\textbf{adj } A}$.

Remark

The determinant of a singular matrix is 0 and so a singular matrix has no inverse.

Example 1.2

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular, find A^{-1}.

Solution

We first find adj A. By definition, we get $\text{adj } A = \begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Since A is non-singular, $|A| = ad - bc \neq 0$.

As $A^{-1} = \frac{1}{|A|}\text{adj } A$, we get $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 1.3

Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 2(7) + (-12) + 3(-1) = -1 \neq 0$.

Therefore, A^{-1} exists. Now, we get

$$\text{adj}\,A = \begin{bmatrix} +\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} & +\begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ -3 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ -5 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ -5 & 3 \end{vmatrix} \end{bmatrix}^T = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}.$$

$$\text{Hence, } A^{-1} = \frac{1}{|A|}(\text{adj}\,A) = \frac{1}{(-1)}\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}.$$

1.2.3 Properties of inverses of matrices

We state and prove some theorems on non-singular matrices.

Theorem 1.4

If A is non-singular, then

(i) $\left|A^{-1}\right| = \dfrac{1}{|A|}$ (ii) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ (iii) $(\lambda A)^{-1} = \dfrac{1}{\lambda}A^{-1}$, where λ is a non-zero scalar.

Proof

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists. By definition,

$$AA^{-1} = A^{-1}A = I_n. \qquad \ldots(1)$$

(i) By (1), we get $\left|AA^{-1}\right| = \left|A^{-1}A\right| = |I_n|$.

Using the product rule for determinants, we get $|A|\left|A^{-1}\right| = |I_n| = 1$.

Hence, $\left|A^{-1}\right| = \dfrac{1}{|A|}$.

(ii) From (1), we get $\left(AA^{-1}\right)^T = \left(A^{-1}A\right)^T = \left(I_n\right)^T$.

Using the reversal law of transpose, we get $\left(A^{-1}\right)^T A^T = A^T\left(A^{-1}\right)^T = I_n$. Hence $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

(iii) Since λ is a non-zero scalar, from (1), we get $(\lambda A)\left(\dfrac{1}{\lambda}A^{-1}\right) = \left(\dfrac{1}{\lambda}A^{-1}\right)(\lambda A) = I_n$.

So, $(\lambda A)^{-1} = \dfrac{1}{\lambda}A^{-1}$.

Theorem 1.5 (Left Cancellation Law)

Let A, B, and C be square matrices of order n. If A is non-singular and $AB = AC$, then $B = C$.

Proof

Since A is non-singular, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$. Taking $AB = AC$ and pre-multiplying both sides by A^{-1}, we get $A^{-1}(AB) = A^{-1}(AC)$. By using the associative property of matrix multiplication and property of inverse matrix, we get $B = C$.

Theorem1.6 (Right Cancellation Law)

Let $A, B,$ and C be square matrices of order n. If A is non-singular and $BA = CA$, then $B = C$.

Proof

Since A is non-singular, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$. Taking $BA = CA$ and post-multiplying both sides by A^{-1}, we get $(BA)A^{-1} = (CA)A^{-1}$. By using the associative property of matrix multiplication and property of inverse matrix, we get $B = C$. ■

Note

If A is singular and $AB = AC$ or $BA = CA$, then B and C need not be equal. For instance, consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

We note that $|A| = 0$ and $AB = AC$; but $B \neq C$.

Theorem 1.7 (Reversal Law for Inverses)

If A and B are non-singular matrices of the same order, then the product AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

Assume that A and B are non-singular matrices of same order n. Then, $|A| \neq 0$, $|B| \neq 0$, both A^{-1} and B^{-1} exist and they are of order n. The products AB and $B^{-1}A^{-1}$ can be found and they are also of order n. Using the product rule for determinants, we get $|AB| = |A||B| \neq 0$. So, AB is non-singular and

$$(AB)(B^{-1}A^{-1}) = (A(BB^{-1}))A^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n;$$
$$(B^{-1}A^{-1})(AB) = (B^{-1}(A^{-1}A))B = (B^{-1}I_n)B = B^{-1}B = I_n.$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$. ■

Theorem 1.8 (Law of Double Inverse)

If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$.

Proof

Assume that A is non-singular. Then $|A| \neq 0$, and A^{-1} exists.

Now $|A^{-1}| = \dfrac{1}{|A|} \neq 0 \Rightarrow A^{-1}$ is also non-singular, and $AA^{-1} = A^{-1}A = I$.

Now, $AA^{-1} = I \Rightarrow \left(AA^{-1}\right)^{-1} = I \Rightarrow \left(A^{-1}\right)^{-1}A^{-1} = I$. ... (1)

Post-multiplying by A on both sides of equation (1), we get $\left(A^{-1}\right)^{-1} = A$. ■

Theorem 1.9

If A is a non-singular square matrix of order n, then

(i) $(\text{adj } A)^{-1} = \text{adj}\left(A^{-1}\right) = \frac{1}{|A|}A$ (ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$ (iv) $\text{adj}(\lambda A) = \lambda^{n-1}\text{adj}(A)$, λ is a nonzero scalar

(v) $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$ (vi) $(\text{adj } A)^T = \text{adj}\left(A^T\right)$

Proof

Since A is a non-singular square matrix, we have $|A| \neq 0$ and so, we get

(i) $A^{-1} = \frac{1}{|A|}(\text{adj } A) \Rightarrow \text{adj } A = |A| A^{-1} \Rightarrow (\text{adj } A)^{-1} = \left(|A| A^{-1}\right)^{-1} = \frac{1}{|A|}\left(A^{-1}\right)^{-1} = \frac{1}{|A|}A$.

Replacing A by A^{-1} in $\text{adj } A = |A| A^{-1}$, we get $\text{adj}\left(A^{-1}\right) = \left|A^{-1}\right|\left(A^{-1}\right)^{-1} = \frac{1}{|A|}A$.

Hence, we get $(\text{adj } A)^{-1} = \text{adj}\left(A^{-1}\right) = \frac{1}{|A|}A$.

(ii) $A(\text{adj } A) = (\text{adj } A)A = |A| I_n \Rightarrow \det\left(A(\text{adj } A)\right) = \det\left((\text{adj } A)A\right) = \det\left(|A| I_n\right)$

$$\Rightarrow |A||\text{adj } A| = |A|^n \Rightarrow |\text{adj } A| = |A|^{n-1}.$$

(iii) For any non-singular matrix B of order n, we have $B(\text{adj } B) = (\text{adj } B)B = |B| I_n$.

Put $B = \text{adj } A$. Then, we get $(\text{adj } A)(\text{adj}(\text{adj } A)) = |\text{adj } A| I_n$.

So, since $|\text{adj } A| = |A|^{n-1}$, we get $(\text{adj } A)(\text{adj}(\text{adj } A)) = |A|^{n-1} I_n$.

Pre-multiplying both sides by A, we get $A\left((\text{adj } A)(\text{adj } (\text{adj } A))\right) = A\left(|A|^{n-1} I_n\right)$.

Using the associative property of matrix multiplication, we get
$\left(A(\text{adj } A)\right)\text{adj}(\text{adj } A) = A\left(|A|^{n-1} I_n\right)$.

Hence, we get $\left(|A| I_n\right)\left(\text{adj}(\text{adj } A)\right) = |A|^{n-1} A$. That is, $\text{adj}(\text{adj } A) = |A|^{n-2} A$.

(iv) Replacing A by λA in $\text{adj}(A) = |A| A^{-1}$ where λ is a non-zero scalar, we get

$$\text{adj}(\lambda A) = |\lambda A|(\lambda A)^{-1} = \lambda^n |A| \frac{1}{\lambda} A^{-1} = \lambda^{n-1}|A| A^{-1} = \lambda^{n-1}\text{adj}(A).$$

(v) By (iii), we have $\text{adj}(\text{adj } A) = |A|^{n-2} A$. So, by taking determinant on both sides, we get

$$|\text{adj}(\text{adj}A)| = \left||A|^{n-2} A\right| = \left(|A|^{(n-2)}\right)^n |A| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}.$$

(vi) Replacing A by A^T in $A^{-1} = \frac{1}{|A|}\text{adj } A$, we get $\left(A^T\right)^{-1} = \frac{1}{|A^T|}\text{adj}\left(A^T\right)$ and hence, we

get $\text{adj}\left(A^T\right) = |A^T| \left(A^T\right)^{-1} = |A| \left(A^{-1}\right)^T = \left(|A| A^{-1}\right)^T = \left(|A| \frac{1}{|A|}\text{adj } A\right)^T = (\text{adj } A)^T$. ■

Note

If A is a non-singular matrix of order 3, then, $|A| \neq 0$. By theorem 1.9 (*ii*), we get $|\text{adj}A| = |A|^2$ and so, $|\text{adj } A|$ is positive. Then, we get $|A| = \pm\sqrt{|\text{adj } A|}$.

So, we get $A^{-1} = \pm\dfrac{1}{\sqrt{|\text{adj } A|}}\text{adj } A$.

Further, by property (*iii*), we get $A = \dfrac{1}{|A|}\text{adj}\left(\text{adj } A\right)$.

Hence, if A is a non-singular matrix of order 3, then we get $A = \pm\dfrac{1}{\sqrt{|\text{adj } A|}}\text{adj}\left(\text{adj } A\right)$.

Example 1.4

If A is a non-singular matrix of odd order, prove that $|\text{adj } A|$ is positive.

Solution

Let A be a non-singular matrix of order $2m+1$, where $m = 0,1,2,\cdots$. Then, we get $|A| \neq 0$ and, by theorem 1.9 (ii), we have $|\text{adj } A| = |A|^{(2m+1)-1} = |A|^{2m}$.

Since $|A|^{2m}$ is always positive, we get that $|\text{adj } A|$ is positive. ■

Example 1.5

Find a matrix A if $\text{adj}(A) = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$.

Solution

First, we find $|\text{adj}(A)| = \begin{vmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix} = 7(77-35) - 7(-7-77) - 7(-5-121) = 1764 > 0$.

So, we get

$$A = \pm\frac{1}{\sqrt{|\text{adj } A|}}\text{adj}\left(\text{adj } A\right) = \pm\frac{1}{\sqrt{1764}}\begin{bmatrix} +(77-35) & -(-7-77) & +(-5-121) \\ -(49+35) & +(49+77) & -(35-77) \\ +(49+77) & -(49-7) & +(77+7) \end{bmatrix}^T$$

$$= \pm\frac{1}{42}\begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}^T = \pm\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}.$$ ■

Example 1.6

If $\text{adj } A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1}.

Solution

$$\text{We compute } |\text{adj } A| = \begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 9 .$$

$$\text{So, we get } A^{-1} = \pm\frac{1}{\sqrt{|\text{adj}(A)|}}\text{adj}(A) = \pm\frac{1}{\sqrt{9}}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \pm\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}.$$

Example1.7

If A is symmetric, prove that adj A is also symmetric.

Solution

Suppose A is symmetric. Then, $A^T = A$ and so, by theorem 1.9 (vi), we get

$\text{adj}\left(A^T\right) = \left(\text{adj } A\right)^T \Rightarrow \text{adj } A = \left(\text{adj } A\right)^T \Rightarrow \text{adj } A$ is symmetric.

Theorem 1.10

If A and B are any two non-singular square matrices of order n, then

$$\text{adj}(AB) = (\text{adj } B)(\text{adj } A).$$

Proof

Replacing A by AB in $\text{adj}(A) = |A|A^{-1}$, we get

$$\text{adj}(AB) = |AB|(AB)^{-1} = \left(|B|B^{-1}\right)\left(|A|A^{-1}\right) = \text{adj}(B)\,\text{adj}(A).$$

Example 1.8

Verify the property $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$.

Solution

For the given A, we get $|A| = (2)(7)-(9)(1) = 14-9 = 5$. So, $A^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{7}{5} & -\frac{9}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$.

$$\text{Then, } \left(A^{-1}\right)^T = \begin{bmatrix} \frac{7}{5} & -\frac{1}{5} \\ -\frac{9}{5} & \frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}. \qquad \text{... (1)}$$

For the given A, We get $A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$. So $\left|A^T\right| = (2)(7)-(1)(9) = 5$.

$$\text{Then, } \left(A^T\right)^{-1} = \frac{1}{5}\begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}. \qquad \text{... (2)}$$

From (1) and (2), we get $\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$. Thus, we have verified the given property.

Example 1.9

Verify $(AB)^{-1}=B^{-1}A^{-1}$ with $A=\begin{bmatrix}0 & -3\\ 1 & 4\end{bmatrix}, B=\begin{bmatrix}-2 & -3\\ 0 & -1\end{bmatrix}$.

Solution

We get $AB = \begin{bmatrix}0 & -3\\ 1 & 4\end{bmatrix}\begin{bmatrix}-2 & -3\\ 0 & -1\end{bmatrix}=\begin{bmatrix}0+0 & 0+3\\ -2+0 & -3-4\end{bmatrix}=\begin{bmatrix}0 & 3\\ -2 & -7\end{bmatrix}$

$$(AB)^{-1} = \frac{1}{(0+6)}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix} \qquad \ldots (1)$$

$$A^{-1} = \frac{1}{(0+3)}\begin{bmatrix}4 & 3\\ -1 & 0\end{bmatrix}=\frac{1}{3}\begin{bmatrix}4 & 3\\ -1 & 0\end{bmatrix}$$

$$B^{-1} = \frac{1}{(2-0)}\begin{bmatrix}-1 & 3\\ 0 & -2\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1 & 3\\ 0 & -2\end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix}-1 & 3\\ 0 & -2\end{bmatrix}\frac{1}{3}\begin{bmatrix}4 & 3\\ -1 & 0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}-7 & -3\\ 2 & 0\end{bmatrix}. \qquad \ldots (2)$$

As the matrices in (1) and (2) are same, $(AB)^{-1}=B^{-1}A^{-1}$ is verified. ■

Example 1.10

If $A=\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}$, find x and y such that $A^2+xA+yI_2=O_2$. Hence, find A^{-1}.

Solution

$$\text{Since } A^2 = \begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}=\begin{bmatrix}22 & 27\\ 18 & 31\end{bmatrix},$$

$$A^2+xA+yI_2=O_2 \Rightarrow \begin{bmatrix}22 & 27\\ 18 & 31\end{bmatrix}+x\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}+y\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}22+4x+y & 27+3x\\ 18+2x & 31+5x+y\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}.$$

So, we get $22+4x+y=0, 31+5x+y=0, 27+3x=0$ and $18+2x=0$.

Hence $x=-9$ and $y=14$. Then, we get $A^2-9A+14I_2=O_2$.

Post-multiplying this equation by A^{-1}, we get $A-9I_2+14A^{-1}=O_2$. Hence, we get

$$A^{-1}=\frac{1}{14}(9I_2-A)=\frac{1}{14}\left(9\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}4 & 3\\ 2 & 5\end{bmatrix}\right)=\frac{1}{14}\begin{bmatrix}5 & -3\\ -2 & 4\end{bmatrix}.$$

■

1.2.4 Application of matrices to Geometry

There is a special type of non-singular matrices which are widely used in applications of matrices to geometry. For simplicity, we consider two-dimensional analytical geometry.

Let O be the origin, and $x'Ox$ and $y'Oy$ be the x-axis and y-axis. Let P be a point in the plane whose coordinates are (x, y) with respect to the coordinate system. Suppose that we rotate the x-axis and y-axis about the origin, through an angle θ as shown in the figure. Let $X'OX$ and $Y'OY$ be the new X-axis and new Y-axis. Let (X, Y) be the new set of coordinates of P with respect to the new coordinate system. Referring to Fig.1.1, we get

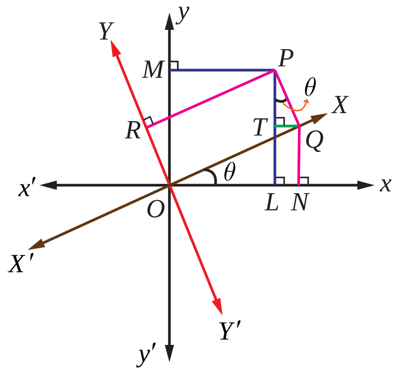

Fig.1.1

$$x = OL = ON - LN = X\cos\theta - QT = X\cos\theta - Y\sin\theta,$$

$$y = PL = PT + TL = QN + PT = X\sin\theta + Y\cos\theta.$$

These equations provide transformation of one coordinate system into another coordinate system. The above two equations can be written in the matrix form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}.$$

Let $W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \end{bmatrix} = W\begin{bmatrix} X \\ Y \end{bmatrix}$ and $|W| = \cos^2\theta + \sin^2\theta = 1$.

So, W has inverse and $W^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. We note that $W^{-1} = W^T$. Then, we get the inverse transformation by the equation

$$\begin{bmatrix} X \\ Y \end{bmatrix} = W^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

Hence, we get the transformation $X = x\cos\theta - y\sin\theta$, $Y = x\sin\theta + y\cos\theta$.

This transformation is used in **Computer Graphics** and determined by the matrix $W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. We note that the matrix W satisfies a special property $W^{-1} = W^T$; that is, $WW^T = W^TW = I$.

Definition 1.3

A square matrix A is called **orthogonal** if $AA^T = A^TA = I$.

Note

A is **orthogonal** if and only if A is non-singular and $A^{-1} = A^T$.

Example 1.11

Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Then, $A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

So, we get

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 .$$

Similarly, we get $A^T A = I_2$. Hence $AA^T = A^T A = I_2 \Rightarrow A$ is orthogonal.

Example 1.12

If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c, and hence A^{-1}.

Solution

If A is orthogonal, then $AA^T = A^T A = I_3$. So, we have

$$AA^T = I_3 \Rightarrow \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 45 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 40 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & c^2 + 13 \end{bmatrix} = 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \left\{\begin{array}{l} 45 + a^2 = 49 \\ b^2 + 40 = 49 \\ c^2 + 13 = 49 \\ 6b + 6 + 6a = 0 \\ 12 - 3c + 3a = 0 \\ 2b - 2c + 18 = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a^2 = 4, b^2 = 9, c^2 = 36, \\ a + b = -1, a - c = -4, b - c = -9 \end{array}\right\} \Rightarrow a = 2, b = -3, c = 6$$

So, we get $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$ and hence, $A^{-1} = A^T = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$.

1.2.5 Application of matrices to Cryptography

One of the important applications of inverse of a non-singular square matrix is in cryptography. Cryptography is an art of communication between two people by keeping the information not known to others. It is based upon two factors, namely encryption and decryption. **Encryption** means the process of transformation of an information (plain form) into an unreadable form (coded form). On the other hand, **Decryption** means the transformation of the coded message back into original form. Encryption and decryption require a secret technique which is known only to the sender and the receiver.

This secret is called a **key**. One way of generating a key is by using a non-singular matrix to encrypt a message by the sender. The receiver decodes (decrypts) the message to retrieve the original message by using the inverse of the matrix. The matrix used for encryption is called **encryption matrix** (**encoding matrix**) and that used for decoding is called **decryption matrix** (**decoding matrix**).

We explain the process of encryption and decryption by means of an example.

Suppose that the sender and receiver consider messages in alphabets $A-Z$ only, both assign the numbers 1-26 to the letters $A-Z$ respectively, and the number 0 to a blank space. For simplicity, the sender employs a key as post-multiplication by a non-singular matrix of order 3 of his own choice. The receiver uses post-multiplication by the inverse of the matrix which has been chosen by the sender.

Let the encoding matrix be

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}.$$

Let the message to be sent by the sender be **"WELCOME"**.

Since the key is taken as the operation of post-multiplication by a square matrix of order 3, the message is cut into pieces **(WEL), (COM), (E)**, each of length 3, and converted into a sequence of row matrices of numbers:

[23 5 12],[3 15 13],[5 0 0].

Note that, we have included two zeros in the last row matrix. The reason is to get a row matrix with 5 as the first entry.

Next, we encode the message by post-multiplying each row matrix as given below:

Uncoded row matrix	Encoding matrix	Coded row matrix

$$\begin{bmatrix}23 & 5 & 12\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}45 & -28 & 23\end{bmatrix};$$

$$\begin{bmatrix}3 & 15 & 13\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}46 & -18 & 3\end{bmatrix};$$

$$\begin{bmatrix}5 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\ 2 & -1 & 0\\ 1 & 0 & 0\end{bmatrix}=\begin{bmatrix}5 & -5 & 5\end{bmatrix}.$$

So the encoded message is $[45\ \ -28\ \ 23]\ [46\ \ -18\ \ 3]\ [5\ \ -5\ \ 5]$

The receiver will decode the message by the reverse key, post-multiplying by the inverse of A.

So the decoding matrix is

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

The receiver decodes the coded message as follows:

Coded row matrix	Decoding matrix	Decoded row matrix
$[45\ \ -28\ \ 23]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$	$= [23\ \ 5\ \ 12]$;
$[46\ \ -18\ \ 3]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$	$= [3\ 15\ 13]$;
$[5\ \ -5\ \ 5]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$	$= [5\ \ 0\ \ 0]$.

So, the sequence of decoded row matrices is $[23\ 5\ 12], [3\ 15\ 13], [5\ 0\ 0]$.

Thus, the receiver reads the message as **"WELCOME"**.

EXERCISE 1.1

1. Find the adjoint of the following:

   (i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

2. Find the inverse (if it exists) of the following:

   (i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$, show that $[F(\alpha)]^{-1} = F(-\alpha)$.

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that $A^2 - 3A - 7I_2 = O_2$. Hence find A^{-1}.

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.

6. If $A=\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A)=(\text{adj } A)A=|A|I_2$.

7. If $A=\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$, verify that $(AB)^{-1}=B^{-1}A^{-1}$.

8. If $\text{adj}(A)=\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

9. If $\text{adj}(A)=\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}.

10. Find $\text{adj}(\text{adj}(A))$ if $\text{adj } A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

11. $A=\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^T A^{-1}=\begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}=\begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

13. Given $A=\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that $AXB=C$.

14. If $A=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1}=\frac{1}{2}\left(A^2-3I\right)$.

15. Decrypt the received encoded message $\begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 20 & 4 \end{bmatrix}$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1-26 to the letters $A-Z$ respectively, and the number 0 to a blank space.

1.3 Elementary Transformations of a Matrix

A matrix can be transformed to another matrix by certain operations called elementary row operations and elementary column operations.

1.3.1 Elementary row and column operations

Elementary row (column) operations on a matrix are as follows:

(i) The interchanging of any two rows (columns) of the matrix

(ii) Replacing a row (column) of the matrix by a non-zero scalar multiple of the row (column) by a non-zero scalar.

(iii) Replacing a row (column) of the matrix by a sum of the row (column) with a non-zero scalar multiple of another row (column) of the matrix.

Elementary row operations and elementary column operations on a matrix are known as elementary transformations.

We use the following notations for elementary row transformations:

(i) Interchanging of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$.

(ii) The multiplication of each element of i^{th} row by a non-zero constant λ is denoted by $R_i \to \lambda R_i$.

(iii) Addition to i^{th} row, a non-zero constant λ multiple of j^{th} row is denoted by $R_i \to R_i + \lambda R_j$.

Similar notations are used for elementary column transformations.

Definition 1.4

Two matrices A and B of same order are said to be equivalent to one another if one can be obtained from the other by the applications of elementary transformations. Symbolically, we write $A \sim B$ to mean that the matrix A is equivalent to the matrix B.

For instance, let us consider a matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$.

After performing the elementary row operation $R_2 \to R_2 + R_1$ on A, we get a matrix B in which the second row is the sum of the second row in A and the first row in A.

Thus, we get $A \sim B = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 5 \\ 1 & -1 & -4 \end{bmatrix}$.

The above elementary row transformation is also represented as follows:

$$\begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 5 \\ 1 & -1 & -4 \end{bmatrix}.$$

Note

An elementary transformation transforms a given matrix into another matrix which need not be equal to the given matrix.

1.3.2 Row-Echelon form

Using the row elementary operations, we can transform a given non-zero matrix to a simplified form called a **Row-echelon form**. In a row-echelon form, we may have rows all of whose entries are zero. Such rows are called **zero rows**. A non-zero row is one in which at least one of the entries is not zero. For instance, in the matrix $\begin{bmatrix} 6 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, R_1 and R_2 are non-zero rows and R_3 is a zero row.

Definition 1.5

A non-zero matrix E is said to be in a **row-echelon form** if:

(i) All zero rows of E occur below every non-zero row of E.

(ii) The first non-zero element in any row i of E occurs in the j^{th} column of E, then all other entries in the j^{th} column of E below the first non-zero element of row i are zeros.

(iii) The first non-zero entry in the i^{th} row of E lies to the left of the first non-zero entry in $(i+1)^{\text{th}}$ row of E.

Note

A non-zero matrix is in a row-echelon form if all zero rows occur as bottom rows of the matrix, and if the first non-zero element in any lower row occurs to the right of the first non-zero entry in the higher row.

The following matrices are in row-echelon form:(i) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$,(ii) $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Consider the matrix in (i). Go up row by row from the last row of the matrix. The third row is a zero row. The first non-zero entry in the second row occurs in the third column and it lies to the right of the first non-zero entry in the first row which occurs in the second column. So the matrix is in row-echelon form.

Consider the matrix in (ii). Go up row by row from the last row of the matrix. All the rows are non-zero rows. The first non-zero entry in the third row occurs in the fourth column and it occurs to the right of the first non-zero entry in the second row which occurs in the third column. The first non-zero entry in the second row occurs in the third column and it occurs to the right of the first non-zero entry in the first row which occurs in the first column. So the matrix is in row-echelon form.

The following matrices are not in row-echelon form:

(i) $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 3 & -2 \\ 5 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$.

Consider the matrix in (i). In this matrix, the first non-zero entry in the third row occurs in the second column and it is on the left of the first non-zero entry in the second row which occurs in the third column. So the matrix is not in row-echelon form.

Consider the matrix in (ii). In this matrix, the first non-zero entry in the second row occurs in the first column and it is on the left of the first non-zero entry in the first row which occurs in the second column. So the matrix is not in row-echelon form.

Method to reduce a matrix $\left[a_{ij}\right]_{m \times n}$ to a row-echelon form.

Step 1

Inspect the first row. If the first row is a zero row, then the row is interchanged with a non-zero row below the first row. If a_{11} is not equal to 0, then go to step 2. Otherwise, interchange the first row R_1 with any other row below the first row which has a non-zero element in the first column; if no row below the first row has non-zero entry in the first column, then consider a_{12}. If a_{12} is not equal to 0, then go to step 2. Otherwise, interchange the first row R_1 with any other row below the first row which has a non-zero element in the second column; if no row below the first row has non-zero entry in the second column, then consider a_{13}. Proceed in the same way till we get a non-zero entry in the first row. This is called **pivoting** and the first non-zero element in the first row is called the **pivot** of the first row.

Step 2

Use the first row and elementary row operations to transform all elements under the pivot to become zeros.

Step 3

Consider the next row as first row and perform steps 1 and 2 with the rows below this row only. Repeat the step until all rows are exhausted.

Example 1.13

Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

Solution

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1,} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2/8} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ This is also a row-echelon form of the given matrix.

So, a row-echelon form of a matrix is not **necessarily** unique.

Example 1.14

Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to a row-echelon form.

Solution

$$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 4R_1} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{2}{3}R_2} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & \frac{22}{3} & 16 \end{bmatrix} \xrightarrow{R_3 \to 3R_3} \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{bmatrix}.$$

1.3.3 Rank of a Matrix

To define the rank of a matrix, we have to know about sub-matrices and minors of a matrix.

Let A be a given matrix. A matrix obtained by deleting some rows and some columns of A is called a **sub-matrix** of A. A matrix is a sub-matrix of itself because it is obtained by leaving zero number of rows and zero number of columns.

Recall that the determinant of a square sub-matrix of a matrix is called a minor of the matrix.

Definition 1.6

The **rank of a matrix** A is defined as the order of a highest order non-vanishing minor of the matrix A. It is denoted by the symbol $\rho(A)$. The rank of a zero matrix is defined to be 0.

Note

(i) If a matrix contains at-least one non-zero element, then $\rho(A) \geq 1$.

(ii) The rank of the identity matrix I_n is n.

(iii) If the rank of a matrix A is r, then there exists at-least one minor of A of order r which does not vanish and every minor of A of order $r+1$ and higher order (if any) vanishes.

(iv) If A is an $m \times n$ matrix, then $\rho(A) \leq \min\{m, n\}$ = minimum of m, n.

(v) A square matrix A of order n has inverse if and only if $\rho(A) = n$.

Example 1.15

Find the rank of each of the following matrices: (i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

Solution

(i) Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$. Then A is a matrix of order 3×3. So $\rho(A) \leq \min\{3, 3\} = 3$. The highest order of minors of A is 3. There is only one third order minor of A.

It is $\begin{vmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{vmatrix} = 3(6-6) - 2(6-6) + 5(3-3) = 0$. So, $\rho(A) < 3$.

Next consider the second-order minors of A.

We find that the second order minor $\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0$. So $\rho(A) = 2$.

(ii) Let $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$. Then A is a matrix of order 3×4. So $\rho(A) \leq \min\{3, 4\} = 3$.

The highest order of minors of A is 3. We search for a non-zero third-order minor of A. But we find that all of them vanish. In fact, we have

$$\begin{vmatrix} 4 & 3 & 1 \\ -3 & -1 & -2 \\ 6 & 7 & -1 \end{vmatrix} = 0; \begin{vmatrix} 4 & 3 & -2 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = 0; \begin{vmatrix} 4 & 1 & -2 \\ -3 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 0; \begin{vmatrix} 3 & 1 & -2 \\ -1 & -2 & 4 \\ 7 & -1 & 2 \end{vmatrix} = 0.$$

So, $\rho(A) < 3$. Next, we search for a non-zero second-order minor of A.

We find that $\begin{vmatrix} 4 & 3 \\ -3 & -1 \end{vmatrix} = -4 + 9 = 5 \neq 0$. So, $\rho(A) = 2$. ■

Remark

Finding the rank of a matrix by searching a highest order non-vanishing minor is quite tedious when the order of the matrix is quite large. There is another easy method for finding the rank of a matrix even if the order of the matrix is quite high. This method is by computing the rank of an equivalent row-echelon form of the matrix. If a matrix is in row-echelon form, then all entries below the **leading diagonal** (it is the line joining the positions of the diagonal elements $a_{11}, a_{22}, a_{33}, \cdots$ of the matrix) are zeros. So, checking whether a minor is zero or not, is quite simple.

Example 1.16

Find the rank of the following matrices which are in row-echelon form :

(i) $\begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution

(i) Let $A = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then A is a matrix of order 3×3 and $\rho(A) \le 3$

The third order minor $|A| = \begin{vmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (2)(3)(1) = 6 \ne 0$. So, $\rho(A) = 3$.

Note that there are three non-zero rows.

(ii) Let $A = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then A is a matrix of order 3×3 and $\rho(A) \le 3$.

The only third order minor is $|A| = \begin{vmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = (-2)(5)(0) = 0$. So $\rho(A) \le 2$.

There are several second order minors. We find that there is a second order minor, for example, $\begin{vmatrix} -2 & 2 \\ 0 & 5 \end{vmatrix} = (-2)(5) = -10 \ne 0$. So, $\rho(A) = 2$.

Note that there are two non-zero rows. The third row is a zero row.

(iii) Let $A = \begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then A is a matrix of order 4×3 and $\rho(A) \le 3$.

The last two rows are zero rows. There are several second order minors. We find that there is a second order minor, for example, $\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = (6)(2) = 12 \ne 0$. So, $\rho(A) = 2$.

Note that there are two non-zero rows. The third and fourth rows are zero rows.

We observe from the above example that the rank of a matrix in row echelon form is equal to the number of non-zero rows in it. We state this observation as a theorem without proof. ■

Theorem 1.11

The rank of a matrix in row echelon form is the number of non-zero rows in it.

The rank of a matrix which is not in a row-echelon form, can be found by applying the following result which is stated without proof.

Theorem 1.12

The rank of a non-zero matrix is equal to the number of non-zero rows in a row-echelon form of the matrix.

Example 1.17

Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by reducing it to a row-echelon form.

Solution

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$. Applying elementary row operations, we get

$$A \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last equivalent matrix is in row-echelon form. It has two non-zero rows. So, $\rho(A) = 2$. ■

Example 1.18

Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to an echelon form.

Solution

Let A be the matrix. Performing elementary row operations, we get

$$A = \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix} \xrightarrow{R_2 \to 2R_2} \begin{bmatrix} 2 & -2 & 4 & 3 \\ -6 & 8 & -4 & -2 \\ 6 & 2 & -1 & 7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix}.$$

$$\xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & -45 & -30 \end{bmatrix} \xrightarrow{R_3 \to R_3 \div (-15)} \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & 3 & 2 \end{bmatrix}.$$

The last equivalent matrix is in row-echelon form. It has three non-zero rows. So, $\rho(A) = 3$. ■

Elementary row operations on a matrix can be performed by pre-multiplying the given matrix by a special class of matrices called elementary matrices.

Definition 1.7

An **elementary matrix** is defined as a matrix which is obtained from an identity matrix by applying only one elementary transformation.

Remark

If we are dealing with matrices with three rows, then all elementary matrices are square matrices of order 3 which are obtained by carrying out only one elementary row operations on the unit matrix I_3. Every elementary row operation that is carried out on a given matrix A can be obtained by pre-multiplying A with elementary matrix. Similarly, every elementary column operation that is carried out on a given matrix A can be obtained by post-multiplying $A$with an elementary matrix. In the present chapter, we use elementary row operations only.

For instance, let us consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Suppose that we do the transformation $R_2 \to R_2 + \lambda R_3$ on A, where $\lambda \neq 0$ is a constant. Then, we get

$$A \xrightarrow{R_2 \to R_2 + \lambda R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \lambda a_{31} & a_{22} + \lambda a_{32} & a_{23} + \lambda a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \qquad \text{....(1)}$$

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix, since we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + \lambda R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$.

Pre-multiplying A by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + \lambda a_{31} & a_{22} + \lambda a_{32} & a_{23} + \lambda a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}. \qquad \text{... (2)}$$

From (1) and (2), we get $A \xrightarrow{R_2 \to R_2 + \lambda R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix} A.$

So, the effect of applying the elementary transformation $R_2 \to R_2 + \lambda R_3$ on A is the same as that of pre-multiplying the matrix A with the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{bmatrix}$.

Similarly, we can show that

(i) the effect of applying the elementary transformation $R_2 \leftrightarrow R_3$ on A is the same as that of pre-multiplying the matrix A with the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(ii) the effect of applying the elementary transformation $R_2 \to \lambda R_2$ on A is the same as that of pre-multiplying the matrix A with the elementary matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

We state the following result without proof.

Theorem 1.13

Every non-singular matrix can be transformed to an identity matrix, by a sequence of elementary row operations.

As an illustration of the above theorem, let us consider the matrix $A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}$.

Then, $|A|=12+3=15\neq 0$. So, A is non-singular. Let us transform A into I_2 by a sequence of elementary row operations. First, we search for a row operation to make a_{11} of A as 1. The elementary row operation needed for this is $R_1\to\left(\frac{1}{2}\right)R_1$. The corresponding elementary matrix is $E_1=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}$.

Then, we get $E_1A=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1 & \frac{-1}{2}\\ 3 & 4\end{bmatrix}$.

Next, let us make all elements below a_{11} of E_1A as 0. There is only one element a_{21}.

The elementary row operation needed for this is $R_2\to R_2+(-3)R_1$.

The corresponding elementary matrix is $E_2=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}$.

Then, we get $E_2(E_1A)=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\ 3 & 4\end{bmatrix}=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{11}{2}\end{bmatrix}$.

Next, let us make a_{22} of $E_2(E_1A)$ as 1. The elementary row operation needed for this is $R_2\to\left(\frac{2}{11}\right)R_2$.

The corresponding elementary matrix is $E_3=\begin{bmatrix}1 & 0\\ 0 & \frac{2}{11}\end{bmatrix}$.

Then, we get $E_3(E_2(E_1A))=\begin{bmatrix}1 & 0\\ 0 & \frac{2}{11}\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & \frac{11}{2}\end{bmatrix}=\begin{bmatrix}1 & -\frac{1}{2}\\ 0 & 1\end{bmatrix}$.

Finally, let us find an elementary row operation to make a_{12} of $E_3(E_2(E_1A))$ as 0. The elementary row operation needed for this is $R_1\to R_1+\left(\frac{1}{2}\right)R_2$. The corresponding elementary matrix is $E_4=\begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}$.

Then, we get $E_4\left(E_3\left(E_2\left(E_1A\right)\right)\right)=\begin{bmatrix}1 & \frac{1}{2}\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{2}\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I_2.$

We write the above sequence of elementary transformations in the following manner:

$$A=\begin{bmatrix}2 & -1\\3 & 4\end{bmatrix}\xrightarrow{R_1\to\left(\frac{1}{2}\right)R_1}\begin{bmatrix}1 & \frac{-1}{2}\\3 & 4\end{bmatrix}\xrightarrow{R_2\to R_2+(-3)R_1}\begin{bmatrix}1 & -\frac{1}{2}\\0 & \frac{11}{2}\end{bmatrix}\xrightarrow{R_2\to\left(\frac{2}{11}\right)R_2}\begin{bmatrix}1 & -\frac{1}{2}\\0 & 1\end{bmatrix}\xrightarrow{R_1\to R_1+\left(\frac{1}{2}\right)R_2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Example 1.19

Show that the matrix $\begin{bmatrix}3 & 1 & 4\\2 & 0 & -1\\5 & 2 & 1\end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Solution

Let $A=\begin{bmatrix}3 & 1 & 4\\2 & 0 & -1\\5 & 2 & 1\end{bmatrix}$. Then, $|A|=3(0+2)-1(2+5)+4(4-0)=6-7+16=15\neq 0.$ So, A is non-singular. Keeping the identity matrix as our goal, we perform the row operations sequentially on A as follows:

$$\begin{bmatrix}3 & 1 & 4\\2 & 0 & -1\\5 & 2 & 1\end{bmatrix}\xrightarrow{R_1\to\frac{1}{3}R_1}\begin{bmatrix}1 & \frac{1}{3} & \frac{4}{3}\\2 & 0 & -1\\5 & 2 & 1\end{bmatrix}\xrightarrow{R_2\to R_2-2R_1,\ R_3\to R_3-5R_1}\begin{bmatrix}1 & \frac{1}{3} & \frac{4}{3}\\0 & -\frac{2}{3} & -\frac{11}{3}\\0 & \frac{1}{3} & -\frac{17}{3}\end{bmatrix}\xrightarrow{R_2\to\left(-\frac{3}{2}\right)R_2}\begin{bmatrix}1 & \frac{1}{3} & \frac{4}{3}\\0 & 1 & \frac{11}{2}\\0 & \frac{1}{3} & -\frac{17}{3}\end{bmatrix}$$

$$\xrightarrow{R_1\to R_1-\frac{1}{3}R_2,\ R_3\to R_3-\frac{1}{3}R_2}\begin{bmatrix}1 & 0 & -\frac{1}{2}\\0 & 1 & \frac{11}{2}\\0 & 0 & -\frac{15}{2}\end{bmatrix}\xrightarrow{R_3\to\left(-\frac{2}{15}\right)R_3}\begin{bmatrix}1 & 0 & -\frac{1}{2}\\0 & 1 & \frac{11}{2}\\0 & 0 & 1\end{bmatrix}\xrightarrow{R_1\to R_1+\frac{1}{2}R_3,R_2\to R_2-\frac{11}{2}R_3}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

■

1.3.4 Gauss-Jordan Method

Let A be a non-singular square matrix of order n. Let B be the inverse of A.

Then, we have $AB=BA=I_n$. By the property of I_n, we have $A=I_nA=AI_n$.

Consider the equation $A=I_nA$ …(1)

Since A is non-singular, pre-multiplying by a sequence of elementary matrices (row operations) on both sides of (1), A on the left-hand-side of (1) is transformed to the identity matrix I_n and the same sequence of elementary matrices (row operations) transforms I_n of the right-hand-side of (1) to a matrix B. So, equation (1) transforms to $I_n=BA$. Hence, the inverse of A is B. That is, $A^{-1}=B$.

Note

If $E_1, E_2, \cdots, E_k$ are elementary matrices (row operations) such that $(E_k \cdots E_2 E_1)A = I_n$, then $A^{-1} = E_k \cdots E_2 E_1$.

Transforming a non-singular matrix A to the form I_n by applying elementary row operations, is called **Gauss-Jordan method.** The steps in finding A^{-1} by Gauss-Jordan method are given below:

Step 1

Augment the identity matrix I_n on the right-side of A to get the matrix $[A \mid I_n]$.

Step 2

Obtain elementary matrices (row operations) $E_1, E_2, \cdots, E_k$ such that $(E_k \cdots E_2 E_1)A = I_n$.

Apply $E_1, E_2, \cdots, E_k$ on $[A \mid I_n]$. Then $[(E_k \cdots E_2 E_1)A \mid (E_k \cdots E_2 E_1)I_n]$. That is, $[I_n \mid A^{-1}]$.

Example 1.20

Find the inverse of the non-singular matrix $A = \begin{bmatrix} 0 & 5 \\ -1 & 6 \end{bmatrix}$, by Gauss-Jordan method.

Solution

Applying Gauss-Jordan method, we get

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 0 & 5 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} -1 & 6 & 0 & 1 \\ 0 & 5 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to (-1)R_1} \left[\begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 5 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{5}R_2} \left[\begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 1 & (1/5) & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + 6R_2} \left[\begin{array}{cc|cc} 1 & 0 & (6/5) & -1 \\ 0 & 1 & (1/5) & 0 \end{array}\right].$$

So, we get $A^{-1} = \begin{bmatrix} (6/5) & -1 \\ (1/5) & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6 & -5 \\ 1 & 0 \end{bmatrix}$. ■

Example 1.21

Find the inverse of $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan method.

Solution

Applying Gauss-Jordan method, we get

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & (1/2) & (1/2) & (1/2) & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & (1/2) & (1/2) & (1/2) & 0 & 0 \\ 0 & (1/2) & -(1/2) & -(3/2) & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to 2R_2} \left[\begin{array}{ccc|ccc} 1 & (1/2) & (1/2) & (1/2) & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_2 \to R_2 + R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right].$$

So, $A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

2. Find the rank of the following matrices by row reduction method:

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

3. Find the inverse of each of the following by Gauss-Jordan method:

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

1.4 Applications of Matrices: Solving System of Linear Equations

One of the important applications of matrices and determinants is solving of system of linear equations. Systems of linear equations arise as mathematical models of several phenomena occurring in biology, chemistry, commerce, economics, physics and engineering. For instance, analysis of circuit theory, analysis of input-output models, and analysis of chemical reactions require solutions of systems of linear equations.

1.4.1 Formation of a System of Linear Equations

The meaning of a system of linear equations can be understood by formulating a mathematical model of a simple practical problem.

Three persons A, B and C go to a supermarket to purchase same brands of rice and sugar. Person A buys 5 Kilograms of rice and 3 Kilograms of sugar and pays ₹ 440. Person B purchases 6 Kilograms of rice and 2 Kilograms of sugar and pays ₹ 400. Person C purchases 8 Kilograms of rice and 5 Kilograms of sugar and pays ₹ 720. Let us formulate a mathematical model to compute the price per Kilogram of rice and the price per Kilogram of sugar. Let x be the price in rupees per Kilogram of rice and y be the price in rupees per Kilogram of sugar. Person A buys 5 Kilograms of rice and 3 Kilograms sugar and pays ₹ 440. So, $5x+3y=440$. Similarly, by considering Person B and Person C, we get $6x+2y=400$ and $8x+5y=720$. Hence the mathematical model is to obtain x and y such that

$$5x+3y=440,\ 6x+2y=400,\ 8x+5y=720.$$

Note

In the above example, the values of x and y which satisfy one equation should also satisfy all the other equations. In other words, the equations are to be satisfied by the same values of x and y simultaneously. If such values of x and y exist, then they are said to form a **solution** for the system of linear equations. In the three equations, x and y appear in first degree only. Hence they are said to form a system of linear equations in two unknowns x and y. They are also called **simultaneous linear equations** in two unknowns x and y. The system has three linear equations in two unknowns x and y.

The equations represent three straight lines in two-dimensional analytical geometry.

In this section, we develop methods using matrices to find solutions of systems of linear equations.

1.4.2 System of Linear Equations in Matrix Form

A system of m linear equations in n unknowns is of the following form:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1, \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2, \\ &\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ &a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m, \end{aligned} \qquad \ldots (1)$$

where the coefficients $a_{ij}, i = 1,2,\cdots,m;\ j = 1,2,\cdots,n$ and $b_k, k = 1,2,\cdots,m$ are constants. If all the b_k's are zeros, then the above system is called a **homogeneous system** of linear equations. On the other hand, if at least one of the b_k's is non-zero, then the above system is called a **non-homogeneous system** of linear equations. If there exist values $\alpha_1, \alpha_2, \cdots, \alpha_n$ for $x_1, x_2, \cdots, x_n$ respectively which satisfy every equation of (1), then the ordered $n-$tuple $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is called a **solution** of (1).

The above system (1) can be put in a matrix form as follows:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$ be the $m \times n$ matrix formed by the coefficients of $x_1, x_2, x_3, \cdots, x_n$. The first row of A is formed by the coefficients of $x_1, x_2, x_3, \cdots, x_n$ in the same order in which they occur in the first equation. Likewise, the other rows of A are formed. The first column is formed by the coefficients of x_1 in the m equations in the same order. The other columns are formed in a similar way.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be the $n \times 1$ order column matrix formed by the unknowns $x_1, x_2, x_3, \cdots, x_n$.

Let $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ be the $m \times 1$ order column matrix formed by the right-hand side constants $b_1, b_2, b_3, \cdots, b_m$.

Then we get

$$AX = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & + \cdots & + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & + \cdots & + a_{2n}x_n \\ \cdots \quad \cdots \quad \cdots & \cdots & \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 & + \cdots & + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = B.$$

Then $AX = B$ is a matrix equation involving matrices and it is called the **matrix form** of the system of linear equations (1). The matrix A is called the **coefficient matrix** of the system and the matrix $\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right]$ is called the **augmented matrix** of the system. We denote the augmented matrix by $[A \,|\, B]$.

As an example, the matrix form of the system of linear equations $2x + 3y - 5z + 7 = 0, 7y + 2z - 3x = 17, 6x - 3y - 8z + 24 = 0$ is $\begin{bmatrix} 2 & 3 & -5 \\ -3 & 7 & 2 \\ 6 & -3 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 17 \\ -24 \end{bmatrix}$.

1.4.3 Solution to a System of Linear equations

The meaning of solution to a system of linear equations can be understood by considering the following cases :

Case (i)

Consider the system of linear equations

$$2x - y = 5, \quad \ldots (1)$$
$$x + 3y = 6. \quad \ldots (2)$$

These two equations represent a pair of straight lines in two dimensional analytical geometry (see the Fig. 1.2). Using (1), we get

$$x = \frac{5+y}{2}. \quad \ldots (3)$$

Substituting (3) in (2) and simplifying, we get $y = 1$.

Substituting $y = 1$ in (1) and simplifying, we get $x = 3$.

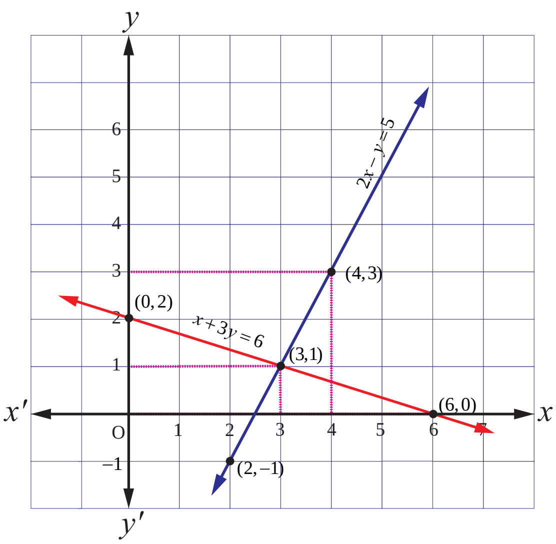

Fig.1.2

Both equations (1) and (2) are satisfied by $x = 3$ and $y = 1$.

That is, **a solution of (1) is also a solution of (2).**

So, we say that the system is consistent and has unique solution $(3,1)$.

The point $(3,1)$ is the point of intersection of the two lines $2x - y = 5$ and $x + 3y = 6$.

Case (ii)

Consider the system of linear equations

$$3x+2y = 5, \quad \text{... (1)}$$

$$6x+4y = 10 \quad \text{... (2)}$$

Using equation (1), we get

$$x = \frac{5-2y}{3} \quad \text{... (3)}$$

Substituting (3) in (2) and simplifying, we get $0=0$.

This informs us that equation (2) is an elementary transformation of equation (1). In fact, by dividing equation (2) by 2, we get equation (1). It is not possible to find uniquely x and y with just a single equation (1).

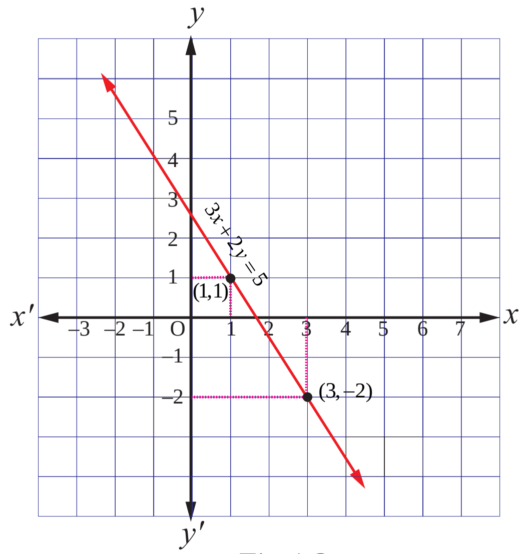

Fig.1.3

So we are forced to assume the value of one unknown, say $y=t$, where t is any real number. Then, $x=\frac{5-2t}{3}$. The two equations (1) and (2) represent one and only one straight line (coincident lines) in two dimensional analytical geometry (see Fig. 1.3). In other words, the system is consistent (a solution of (1) is also a solution of (2)) and has infinitely many solutions, as t can assume any real value.

Case (iii)

Consider the system of linear equations

$$4x+y = 6, \quad \text{... (1)}$$

$$8x+2y = 18. \quad \text{... (2)}$$

Using equation (1), we get

$$x = \frac{6-y}{4} \quad \text{... (3)}$$

Substituting (3) in (2) and simplifying, we get $12=18$.

This is a contradicting result, which informs us that equation (2) is inconsistent with equation (1). So, a solution of (1) is not a solution of (2).

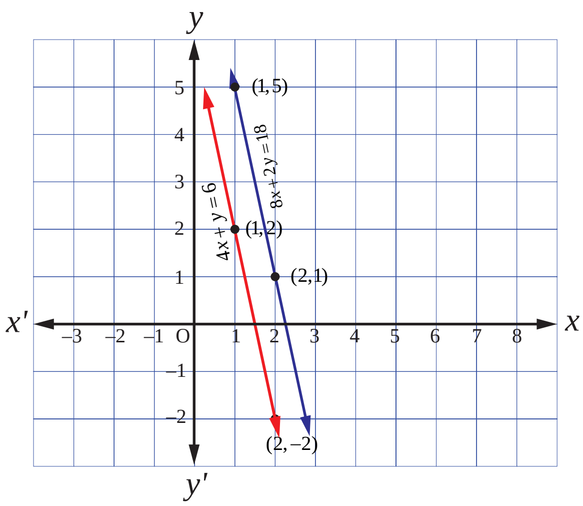

Fig.1.4

In other words, the system is inconsistent and has no solution. We note that the two equations represent two parallel straight lines (non-coincident) in two dimensional analytical geometry (see Fig. 1.4). We know that two non-coincident parallel lines never meet in real points.

Note

(1) Interchanging any two equations of a system of linear equations does not alter the solution of the system.

(2) Replacing an equation of a system of linear equations by a non-zero constant multiple of itself does not alter the solution of the system.

(3) Replacing an equation of a system of linear equations by addition of itself with a non-zero multiple of any other equation of the system does not alter the solution of the system.

Definition 1.8

A system of linear equations having at least one solution is said to be **consistent**. A system of linear equations having no solution is said to be **inconsistent**.

Remark

If the number of the equations of a system of linear equations is equal to the number of unknowns of the system, then the coefficient matrix A of the system is a square matrix. Further, if A is a non-singular matrix, then the solution of system of equations can be found by any one of the following methods : (i) matrix inversion method, (ii) Cramer's rule, (iii) Gaussian elimination method.

1.4.3 (i) Matrix Inversion Method

This method can be applied only when the coefficient matrix is a square matrix and non-singular.

Consider the matrix equation

$$AX = B, \qquad \ldots (1)$$

where A is a square matrix and non-singular. Since A is non-singular, A^{-1} exists and $A^{-1}A = AA^{-1} = I$.

Pre-multiplying both sides of (1) by A^{-1}, we get $A^{-1}(AX) = A^{-1}B$. That is, $(A^{-1}A)X = A^{-1}B$.

Hence, we get $X = A^{-1}B$.

Example 1.22

Solve the following system of linear equations, using matrix inversion method:

$5x + 2y = 3,\ 3x + 2y = 5$.

Solution

The matrix form of the system is $AX = B$, where $A = \begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

We find $|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix} = 10 - 6 = 4 \neq 0$. So, A^{-1} exists and $A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$.

Then, applying the formula $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -4 \\ 16 \end{bmatrix} = \begin{bmatrix} \frac{-4}{4} \\ \frac{16}{4} \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

So the solution is $(x = -1, y = 4)$. ■

Example 1.23

Solve the following system of equations, using matrix inversion method:

$2x_1 + 3x_2 + 3x_3 = 5,\quad x_1 - 2x_2 + x_3 = -4,\quad 3x_1 - x_2 - 2x_3 = 3.$

Solution

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

We find $|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4+1)-3(-2-3)+3(-1+6) = 10+15+15 = 40 \neq 0$.

So, A^{-1} exists and

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{40}\begin{bmatrix} +(4+1) & -(-2-3) & +(-1+6) \\ -(-6+3) & +(-4-9) & -(-2-9) \\ +(3+6) & -(2-3) & +(-4-3) \end{bmatrix}^T = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Then, applying $X = A^{-1}B$, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

So, the solution is $(x_1 = 1, x_2 = 2, x_3 = -1)$.

Example 1.24

If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$.

Solution

We find $AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$

and $BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$.

So, we get $AB = BA = 8I_3$. That is, $\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$. Hence, $B^{-1} = \frac{1}{8}A$.

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}. \text{ That is, } B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}.$$

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$.

Hence, the solution is $(x = 3, y = -2, z = -1)$.

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

   (i) $2x+5y=-2,\ \ x+2y=-3$ (ii) $2x-y=8,\ \ 3x+2y=-2$

   (iii) $2x+3y-z=9,\ x+y+z=9,\ 3x-y-z=-1$

   (iv) $x+y+z-2=0,\ 6x-4y+5z-31=0,\ 5x+2y+2z=13$

2. If $A=\begin{bmatrix}-5 & 1 & 3\\ 7 & 1 & -5\\ 1 & -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1 & 2\\ 3 & 2 & 1\\ 2 & 1 & 3\end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x+y+2z=1, 3x+2y+z=7, 2x+y+3z=2.$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 per month at the end of the first month after 3 years of service and ₹ 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

5. The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn ₹ 15,000, ₹ 1,000 and ₹ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

1.4.3 (ii) Cramer's Rule

This rule can be applied only when the coefficient matrix is a square matrix and non-singular. It is explained by considering the following system of equations:

$$a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1,$$
$$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2,$$
$$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3,$$

where the coefficient matrix $\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ is non-singular. Then $\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}\neq 0.$

Let us put $\Delta=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$. Then, we have

$$x_1\Delta = x_1\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix}a_{11}x_1 & a_{12} & a_{13}\\ a_{21}x_1 & a_{22} & a_{23}\\ a_{31}x_1 & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix}a_{11}x_1+a_{12}x_2+a_{13}x_3 & a_{12} & a_{13}\\ a_{21}x_1+a_{22}x_2+a_{23}x_3 & a_{22} & a_{23}\\ a_{31}x_1+a_{32}x_2+a_{33}x_3 & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix}b_1 & a_{12} & a_{13}\\ b_2 & a_{22} & a_{23}\\ b_3 & a_{32} & a_{33}\end{vmatrix}=\Delta_1$$

Since $\Delta \neq 0$, we get $x_1 = \dfrac{\Delta_1}{\Delta}$.

Similarly, we get $x_2 = \dfrac{\Delta_2}{\Delta}, x_3 = \dfrac{\Delta_3}{\Delta}$, where $\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$.

Thus, we have the Cramer's rule $x_1 = \dfrac{\Delta_1}{\Delta}, x_2 = \dfrac{\Delta_2}{\Delta}, x_3 = \dfrac{\Delta_3}{\Delta}$,

where $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$

Note

Replacing the first column elements a_{11}, a_{21}, a_{31} of Δ with b_1, b_2, b_3 respectively, we get Δ_1.

Replacing the second column elements a_{12}, a_{22}, a_{32} of Δ with b_1, b_2, b_3 respectively, we get Δ_2.

Replacing the third column elements a_{13}, a_{23}, a_{33} of Δ with b_1, b_2, b_3 respectively, we get Δ_3.

If $\Delta = 0$, Cramer's rule cannot be applied.

Example 1.25

Solve, by Cramer's rule, the system of equations

$x_1 - x_2 = 3, 2x_1 + 3x_2 + 4x_3 = 17, x_2 + 2x_3 = 7.$

Solution

First we evaluate the determinants

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 6 \neq 0,\ \Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 12,\ \Delta_2 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = -6,\ \Delta_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix} = 24.$$

By Cramer's rule, we get $x_1 = \dfrac{\Delta_1}{\Delta} = \dfrac{12}{6} = 2,\ x_2 = \dfrac{\Delta_2}{\Delta} = \dfrac{-6}{6} = -1,\ x_3 = \dfrac{24}{6} = 4.$

So, the solution is $(x_1 = 2, x_2 = -1, x_3 = 4)$.

Example 1.26

In a T20 match, a team needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points $(10,8), (20,16), (40,22)$, can you conclude that the team won the match?

Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is $(70, 0)$.)

Solution

The path $y = ax^2 + bx + c$ passes through the points $(10,8), (20,16), (40,22)$. So, we get the system of equations $100a + 10b + c = 8, 400a + 20b + c = 16, 1600a + 40b + c = 22$. To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 1000 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1000[-2+12-16] = -6000,$$

$$\Delta_1 = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 20 \begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix} = 20[-8+3+10] = 100,$$

$$\Delta_2 = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix} = 200 \begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix} = 200[-3+48-84] = -7800,$$

$$\Delta_3 = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix} = 2000 \begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix} = 2000[-10+84-64] = 20000.$$

By Cramer's rule, we get $a = \frac{\Delta_1}{\Delta} = -\frac{1}{60}, b = \frac{\Delta_2}{\Delta} = \frac{7800}{6000} = \frac{78}{60} = \frac{13}{10}, c = \frac{\Delta_3}{\Delta} = -\frac{20000}{6000} = -\frac{20}{6} = -\frac{10}{3}$.

So, the equation of the path is $y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$.

When $x = 70$, we get $y = 6$. So, the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the team won the match.

EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

   (i) $5x - 2y + 16 = 0, x + 3y - 7 = 0$

   (ii) $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

   (iii) $3x + 3y - z = 11,\ 2x - y + 2z = 9,\ 4x + 3y + 2z = 25$

   (iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem).

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of the two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

1.4.3 (iii) Gaussian Elimination Method

This method can be applied even if the coefficient matrix is singular matrix and rectangular matrix. It is essentially the method of substitution which we have already seen. In this method, we transform the augmented matrix of the system of linear equations into row-echelon form and then by back-substitution, we get the solution.

Example 1.27

Solve the following system of linear equations, by Gaussian elimination method :

$4x+3y+6z=25,\ x+5y+7z=13,\ 2x+9y+z=1.$

Solution

Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1,} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div (-1)]{R_2 \to R_2 \div (-1),} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 1 & 13 & 25 \end{array}\right] \xrightarrow{R_3 \to 17R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{array}\right].$$

The equivalent system is written by using the echelon form:

$$x+5y+7z = 13, \qquad \ldots (1)$$

$$17y+22z = 27, \qquad \ldots (2)$$

$$199z = 398. \qquad \ldots (3)$$

From (3), we get $z=\frac{398}{199}=2$.

Substituting $z=2$ in (2), we get $y=\frac{27-22\times 2}{17}=\frac{-17}{17}=-1$.

Substituting $z=2, y=-1$ in (1), we get $x=13-5\times(-1)-7\times 2=4$.

So, the solution is $(x=4, y=-1, z=2)$. ■

Note. The above method of going from the last equation to the first equation is called the **method of back substitution.**

Example 1.28

The upward speed $v(t)$ of a rocket at time t is approximated by $v(t)=at^2+bt+c,\ 0\le t\le 100$ where $a,b,$ and c are constants. It has been found that the speed at times $t=3, t=6$, and $t=9$ seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time $t=15$ seconds. (Use Gaussian elimination method.)

Solution

Since $v(3)=64,\ v(6)=133,$ and $v(9)=208$, we get the following system of linear equations

$$\begin{aligned} 9a+3b+c &= 64\,, \\ 36a+6b+c &= 133\,, \\ 81a+9b+c &= 208\,. \end{aligned}$$

We solve the above system of linear equations by Gaussian elimination method.

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operations, we get

$$[A\,|\,B]=\left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right] \xrightarrow{R_2\to R_2-4R_1,\,R_3\to R_3-9R_1} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right] \xrightarrow{R_2\to R_2\div(-3),\,R_3\to R_3\div(-2)} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 9 & 4 & 184 \end{array}\right]$$

$$\xrightarrow{R_3\to 2R_3} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 18 & 8 & 368 \end{array}\right] \xrightarrow{R_3\to R_3-9R_2} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{R_3\to(-1)R_3} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Writing the equivalent equations from the row-echelon matrix, we get

$9a+3b+c=64,\ 2b+c=41,\ c=1.$

By back substitution, we get $c=1,\ b=\dfrac{(41-c)}{2}=\dfrac{(41-1)}{2}=20,\ a=\dfrac{64-3b-c}{9}=\dfrac{64-60-1}{9}=\dfrac{1}{3}.$

So, we get $v(t)=\dfrac{1}{3}t^2+20t+1$. Hence, $v(15)=\dfrac{1}{3}(225)+20(15)+1=75+300+1=376.$ ■

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

   (i) $2x-2y+3z=2,\quad x+2y-z=3,\quad 3x-y+2z=1$

   (ii) $2x+4y+6z=22,\quad 3x+8y+5z=27,\quad -x+y+2z=2$

2. If ax^2+bx+c is divided by $x+3, x-5,$ and $x-1,$ the remainders are $21, 61$ and 9 respectively. Find a, b and c. (Use Gaussian elimination method.)

3. An amount of ₹ 65,000 is invested in three bonds at the rates of $6\%, 8\%$ and 9% per annum respectively. The total annual income is ₹ 4,800. The income from the third bond is ₹ 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

4. A boy is walking along the path $y=ax^2+bx+c$ through the points $(-6,8), (-2,-12)$, and $(3,8)$. He wants to meet his friend at $P(7,60)$. Will he meet his friend? (Use Gaussian elimination method.)

1.5 Applications of Matrices: Consistency of System of Linear Equations by Rank Method

In section 1.3.3, we have already defined consistency of a system of linear equation. In this section, we investigate it by using rank method. We state the following theorem without proof:

Theorem 1.14 (Rouché-Capelli Theorem)

A system of linear equations, written in the matrix form as $AX = B$, is consistent if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix; that is, $\rho(A) = \rho([A \mid B])$.

We apply the theorem in the following examples.

1.5.1 Non-homogeneous Linear Equations

Example 1.29

Test for consistency of the following system of linear equations and if possible solve: $x+2y-z=3,\ 3x-y+2z=1,\ x-2y+3z=3,\ x-y+z+1=0$.

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

The augmented matrix is $[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{array}\right]$.

Applying Gaussian elimination method on $[A \mid B]$, we get

$$[A \mid B] \xrightarrow[\substack{R_3 \to R_3 - R_1, \\ R_4 \to R_4 - R_1,}]{R_2 \to R_2 - 3R_1,} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & -4 \end{array}\right] \xrightarrow[\substack{R_3 \to (-1)R_3, \\ R_4 \to (-1)R_4}]{R_2 \to (-1)R_2,} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -2 & 4 \end{array}\right]$$

$$\xrightarrow[R_4 \to 7R_4 - 3R_2]{R_3 \to 7R_3 - 4R_2,} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 \div (-8)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There are three non-zero rows in the row-echelon form of $[A \mid B]$. So, $\rho\left([A \mid B]\right) = 3$.

So, the row-echelon form of A is $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There are three non-zero rows in it. So $\rho(A) = 3$.

Hence, $\rho(A) = \rho([A\,|\,B]) = 3$.

From the echelon form, we write the equivalent system of equations

$x + 2y - z = 3,\ 7y - 5z = 8,\ z = 4,\ 0 = 0.$

The last equation $0 = 0$ is meaningful. By the method of back substitution, we get

$$\begin{aligned} z &= 4 & & \\ 7y - 20 &= 8 & \Rightarrow \quad & y = 4, \\ x &= 3 - 8 + 4 & \Rightarrow \quad & x = -1. \end{aligned}$$

So, the solution is $(x = -1, y = 4, z = 4)$. (Note that A is not a square matrix.)

Here the given system is consistent and the solution is unique.

Example 1.30

Test for consistency of the following system of linear equations and if possible solve:

$4x - 2y + 6z = 8,\ x + y - 3z = -1,\ 15x - 3y + 9z = 21.$

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}.$$

Applying elementary row operations on the augmented matrix $[A\,|\,B]$, we get

$$[A\,|\,B] = \left[\begin{array}{ccc|c} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{array}\right] \xrightarrow[R_3 \to R_3 - 15R_1]{R_2 \to R_2 - 4R_1,} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div (-18)]{R_2 \to R_2 \div (-6),} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So, $\rho(A) = \rho([A\,|\,B]) = 2 < 3$. From the echelon form, we get the equivalent equations

$$x + y - 3z = -1,\ y - 3z = -2,\ 0 = 0.$$

The equivalent system has two non-trivial equations and three unknowns. So, one of the unknowns should be fixed at our choice in order to get two equations for the other two unknowns. We fix z arbitrarily as a real number t, and we get $y = 3t - 2,\ x = -1 - (3t - 2) + 3t = 1$. So, the solution is $(x = 1, y = 3t - 2, z = t)$, where t is real. The above solution set is a one-parameter family of solutions. Here, the given system is consistent and has infinitely many solutions which form a one parameter family of solutions.

Note

In the above example, the square matrix A is singular and so matrix inversion method cannot be applied to solve the system of equations. However, Gaussian elimination method is applicable and we are able to decide whether the system is consistent or not. The next example also confirms the supremacy of Gaussian elimination method over other methods.

Example 1.31

Test for consistency of the following system of linear equations and if possible solve: $x-y+z=-9,\ 2x-2y+2z=-18,\ 3x-3y+3z+27=0$.

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX=B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & -3 & 3 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B=\begin{bmatrix} -9 \\ -18 \\ -27 \end{bmatrix}.$$

Applying elementary row operations on the augmented matrix $[A\,|\,B]$, we get

$$[A\,|\,B]=\left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -2 & 2 & -18 \\ 3 & -3 & 3 & -27 \end{array}\right] \xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1,} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So, $\rho(A)=\rho([A\,|\,B])=1<3$.

From the echelon form, we get the equivalent equations $x-y+z=-9,\ \ 0=0,\ 0=0$.
The equivalent system has one non-trivial equation and three unknowns.
Taking $y=s, z=t$ arbitrarily, we get $x-s+t=-9$; or $x=-9+s-t$.
So, the solution is $(x=-9+s-t, y=s, z=t)$, where s and t are parameters.
The above solution set is a two-parameter family of solutions.

Here, the given system of equations is consistent and has infinitely many solutions which form a two parameter family of solutions.

Example 1.32

Test the consistency of the following system of linear equations

$x-y+z=-9,\ 2x-y+z=4,\ 3x-y+z=6, 4x-y+2z=7$.

Solution

Here the number of unknowns is 3.
The matrix form of the system of equations is $AX=B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B=\begin{bmatrix} -9 \\ 4 \\ 6 \\ 7 \end{bmatrix}.$$

Applying elementary row operations on the augmented matrix $[A\,|\,B]$, we get

$$[A\,|\,B]=\left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array}\right] \xrightarrow[R_4\to R_4-4R_1]{R_2\to R_2-2R_1,\ R_3\to R_3-3R_1,} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{array}\right]$$

$$\xrightarrow[R_4\to R_4-3R_2]{R_3\to R_3-2R_2,} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{array}\right] \xrightarrow{R_3\leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{array}\right]$$

So, $\rho(A)=3$ and $\rho([A\,|\,B])=4$. Hence $\rho(A)\neq\rho([A\,|\,B])$.

If we write the equivalent system of equations using the echelon form, we get

$x-y+z=-9,\quad y-z=22,\ z=-23,\quad 0=-11.$

The last equation is a contradiction.

So the given system of equations is inconsistent and has no solution. ■

By Rouché-Capelli theorem, we have the following rule:

- If there are n unknowns in the system of equations and $\rho(A)=\rho([A\,|\,B])=n$, then the system $AX=B$, is consistent and has a unique solution.
- If there are n unknowns in the system $AX=B$, and $\rho(A)=\rho([A\,|\,B])=n-k, k\neq 0$ then the system is consistent and has infinitely many solutions and these solutions form a $k-$parameter family. In particular, if there are 3 unknowns in a system of equations and $\rho(A)=\rho([A\,|\,B])=2$, then the system has infinitely many solutions and these solutions form a one parameter family. In the same manner, if there are 3 unknowns in a system of equations and $\rho(A)=\rho([A\,|\,B])=1$, then the system has infinitely many solutions and these solutions form a two parameter family.
- If $\rho(A)\neq\rho([A\,|\,B])$, then the system $AX=B$ is inconsistent and has no solution.

Example 1.33

Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions: $x+y+z=a,\ x+2y+3z=b,\ 3x+5y+7z=c.$

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX=B$, where $A=\begin{bmatrix}1&1&1\\1&2&3\\3&5&7\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}a\\b\\c\end{bmatrix}.$

Applying elementary row operations on the augmented matrix $[A\,|\,B]$, we get

$$[A\,|\,B]=\left[\begin{array}{ccc|c}1&1&1&a\\1&2&3&b\\3&5&7&c\end{array}\right]\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-R_1,}\left[\begin{array}{ccc|c}1&1&1&a\\0&1&2&b-a\\0&2&4&c-3a\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-2R_2}\left[\begin{array}{ccc|c}1&1&1&a\\0&1&2&b-a\\0&0&0&(c-3a)-2(b-a)\end{array}\right]=\left[\begin{array}{ccc|c}1&1&1&a\\0&1&2&b-a\\0&0&0&(c-2b-a)\end{array}\right].$$

In order that the system should have one parameter family of solutions, we must have $\rho(A)=\rho([A,B])=2$. So, the third row in the echelon form should be a zero row.

So, $c-2b-a=0 \quad\Rightarrow\quad c=a+2b.$ ■

Example 1.34

Investigate for what values of λ and μ the system of linear equations

$x+2y+z=7,\ x+y+\lambda z=\mu, x+3y-5z=5$

has (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$.

Applying elementary row operations on the augmented matrix $[A \mid B]$, we get

$$[A \mid B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1,} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda-1 & \mu-7 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda-7 & \mu-9 \end{array}\right].$$

(i) If $\lambda = 7$ and $\mu \neq 9$, then $\rho(A) = 2$ and $\rho([A \mid B]) = 3$. So $\rho(A) \neq \rho([A \mid B])$. Hence the given system is inconsistent and has no solution.

(ii) If $\lambda \neq 7$ and μ is any real number, then $\rho(A) = 3$ and $\rho([A \mid B]) = 3$.
So $\rho(A) = \rho([A \mid B]) = 3 =$ Number of unknowns. Hence the given system is consistent and has a unique solution.

(iii) If $\lambda = 7$ and $\mu = 9$, then $\rho(A) = 2$ and $\rho([A \mid B]) = 2$.

So, $\rho(A) = \rho([A \mid B]) = 2 <$ Number of unknowns. Hence the given system is consistent and has infinite number of solutions. ■

EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by rank method.
   (i) $x - y + 2z = 2, \quad 2x + y + 4z = 7, \quad 4x - y + z = 4$
   (ii) $3x + y + z = 2, \quad x - 3y + 2z = 1, \quad 7x - y + 4z = 5$
   (iii) $2x + 2y + z = 5, \quad x - y + z = 1, \quad 3x + y + 2z = 4$
   (iv) $2x - y + z = 2, \quad 6x - 3y + 3z = 6, \quad 4x - 2y + 2z = 4$

2. Find the value of k for which the equations $kx - 2y + z = 1, \quad x - 2ky + z = -2, \quad x - 2y + kz = 1$ have
   (i) no solution (ii) unique solution (iii) infinitely many solution

3. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9$, $7x + 3y - 5z = 8, \; 2x + 3y + \lambda z = \mu$, have
   (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

1.5.2 Homogeneous system of linear equations

We recall that a homogeneous system of linear equations is given by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0,\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad &\cdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0,
\end{aligned} \qquad \ldots (1)$$

where the coefficients $a_{ij}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$ are constants. The above system is always satisfied by $x_1 = 0, x_2 = 0, \cdots, x_n = 0.$ This solution is called the **trivial solution** of (1). In other words, the system (1) always possesses a solution.

The above system (1) can be put in the matrix form $AX = O_{m\times 1}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, O_{m\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We will denote $O_{m\times 1}$ simply by the capital letter $O.$ Since O is the zero column matrix, it is always true that $\rho(A) = \rho([A\,|\,O]) \le m.$ **So, by Rouché-Capelli Theorem, any system of homogeneous linear equations is always consistent.**

Suppose that $m < n,$ then there are more number of unknowns than the number of equations. So $\rho(A) = \rho([A\,|\,O]) < n.$ Hence, system (1) possesses a non-trivial solution.

Suppose that $m = n,$ then there are equal number of equations and unknowns:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0,\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0,\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad &\cdots\\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= 0,
\end{aligned} \qquad \ldots (2)$$

Two cases arise.

Case (i)

If $\rho(A) = \rho([A\,|\,O]) = n,$ then system (2) has a **unique solution** and it is the **trivial solution.** Since $\rho(A) = n, |A| \ne 0.$ So for **trivial solution** $|A| \ne 0$.

Case (ii)

If $\rho(A) = \rho([A\,|\,O]) < n,$ then system (2) has a **non-trivial solution**. Since $\rho(A) < n, |A| = 0.$ In other words, **the homogeneous system (2) has a non-trivial solution if and only if the determinant of the coefficient matrix is zero.**

Suppose that $m > n,$ then there are more number of equations than the number of unknowns. Reducing the system by elementary transformations, we get $\rho(A) = \rho([A\,|\,O]) \le n.$

Example 1.35

Solve the following system:

$x+2y+3z=0,\ 3x+4y+4z=0,\ 7x+10y+12z=0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form (Gaussian elimination method), the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1,} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -9 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 \div (-1)]{R_2 \to R_2 \div (-1),} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 4 & 9 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 \div (-1)} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So, $\rho(A) = \rho([A \mid O]) = 3 =$ Number of unknowns.

Hence, the system has a unique solution. Since $x=0,\ y=0,\ z=0$ is always a solution of the homogeneous system, the only solution is the trivial solution $x=0,\ y=0,\ z=0.$ ■

Note

In the above example, we find that

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{vmatrix} = 1(48-40) - 2(36-28) + 3(30-28) = 8 - 16 + 6 = -2 \neq 0.$$

Example 1.36

Solve the system: $x+3y-2z=0,\ 2x-y+4z=0,\ x-11y+14z=0.$

Solution

Here the number of unknowns is 3.

Transforming into echelon form (Gaussian elimination method), the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1,} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 \div (-2)]{R_2 \to R_2 \div (-1),} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 7 & -8 & 0 \\ 0 & 7 & -8 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So, $\rho(A) = \rho([A \mid O]) = 2 < 3 =$ Number of unknowns.

Hence, the system has a one parameter family of solutions.

Writing the equations using the echelon form, we get

$x+3y-2z=0, \qquad 7y-8z=0, \qquad 0=0.$

Taking $z=t$, where t is an arbitrary real number, we get by back substitution,

$$z = t,$$

$$7y - 8t = 0 \Rightarrow y = \frac{8t}{7},$$

$$x + 3\left(\frac{8t}{7}\right) - 2t = 0 \Rightarrow x + \frac{24t - 14t}{7} = 0 \Rightarrow x = -\frac{10t}{7}.$$

So, the solution is $\left(x = -\frac{10t}{7}, y = \frac{8t}{7}, z = t\right)$, where t is any real number.

Example 1.37

Solve the system: $x + y - 2z = 0, 2x - 3y + z = 0, 3x - 7y + 10z = 0, 6x - 9y + 10z = 0.$

Solution

Here the number of equations is 4 and the number of unknowns is 3. Reducing the augmented matrix to echelon-form, we get

$$[A \mid O] = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & -7 & 10 & 0 \\ 6 & -9 & 10 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 - 6R_1]{\substack{R_2 \to R_2 - 2R_1, \\ R_3 \to R_3 - 3R_1,}} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 16 & 0 \\ 0 & -15 & 22 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 \div (-2)]{R_2 \to R_2 \div (-5),} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & -8 & 0 \\ 0 & -15 & 22 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \to R_4 + 15R_2]{R_3 \to R_3 - 5R_2,} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \xrightarrow[R_4 \to R_4 \div 7]{R_3 \to R_3 \div (-3),} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, $\rho(A) = \rho([A \mid O]) = 3 =$ Number of unknowns.

Hence the system has trivial solution only.

Example 1.38

Determine the values of λ for which the following system of equations

$(3\lambda - 8)x + 3y + 3z = 0,\ 3x + (3\lambda - 8)y + 3z = 0,\ 3x + 3y + (3\lambda - 8)z = 0$

has a non-trivial solution.

Solution

Here the number of unknowns is 3. So, if the system is consistent and has a non-trivial solution, then the rank of the coefficient matrix is equal to the rank of the augmented matrix and is less than 3. So the determinant of the coefficient matrix should be 0.

Hence we get

$$\begin{vmatrix} 3\lambda - 8 & 3 & 3 \\ 3 & 3\lambda - 8 & 3 \\ 3 & 3 & 3\lambda - 8 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 3\lambda - 2 & 3\lambda - 2 & 3\lambda - 2 \\ 3 & 3\lambda - 8 & 3 \\ 3 & 3 & 3\lambda - 8 \end{vmatrix} = 0 \text{ (by applying } R_1 \to R_1 + R_2 + R_3)$$

$$\text{or } (3\lambda - 2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3\lambda - 8 & 3 \\ 3 & 3 & 3\lambda - 8 \end{vmatrix} = 0 \text{ (by taking out } (3\lambda - 2) \text{ from } R_1)$$

$$\text{or } (3\lambda-2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3\lambda-11 & 0 \\ 0 & 0 & 3\lambda-11 \end{vmatrix} = 0 \text{ (by applying } R_2 \to R_2 - 3R_1, R_3 \to R_3 - 3R_1)$$

or $(3\lambda-2)(3\lambda-11)^2\ 0$. So $\lambda = \frac{2}{3}$ and $\lambda = \frac{11}{3}$.

We now give an application of system of linear homogeneous equations to chemistry. You are already aware of balancing chemical reaction equations by inspecting the number of atoms present on both sides. A direct method is explained as given below.

Example 1.39

By using Gaussian elimination method, balance the chemical reaction equation:

$C_5H_8 + O_2 \to CO_2 + H_2O.$

(The above is the reaction that is taking place in the burning of organic compound called isoprene.)

Solution

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$$x_1C_5H_8 + x_2O_2 = x_3CO_2 + x_4H_2O. \qquad \text{.. (1)}$$

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogenous equation

$$5x_1 = x_3 \Rightarrow 5x_1 - x_3 = 0. \qquad \text{... (2)}$$

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$$8x_1 = 2x_4 \Rightarrow 4x_1 - x_4 = 0, \qquad \text{... (3)}$$

$$2x_2 = 2x_3 + x_4 \Rightarrow 2x_2 - 2x_3 - x_4 = 0. \qquad \text{... (4)}$$

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns.

The augmented matrix is $[A\,|\,B] = \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$.

By Gaussian elimination method, we get

$$[A\,|\,B] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 4R_3 - 5R_1} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & -4 & 5 & 0 \end{array}\right].$$

Therefore, $\rho(A) = \rho([A\,|\,B]) = 3 < 4 =$ Number of unknowns.

The system is consistent and has infinite number of solutions.

Writing the equations using the echelon form, we get $4x_1 - x_4 = 0, 2x_2 - 2x_3 - x_4 = 0, -4x_3 + 5x_4 = 0.$

So, one of the unknowns should be chosen arbitrarily as a non-zero real number.

Let us choose $x_4 = t, t \neq 0$. Then, by back substitution, we get $x_3 = \frac{5t}{4}, x_2 = \frac{7t}{4}, x_1 = \frac{t}{4}$.

Since x_1, x_2, x_3, and x_4 are positive integers, let us choose $t = 4$.

Then, we get $x_1 = 1, x_2 = 7, x_3 = 5$ and $x_4 = 4$.

So, the balanced equation is $C_5H_8 + 7O_2 \rightarrow 5CO_2 + 4H_2O$.

Example 1.40

If the system of equations $px + by + cz = 0,\ ax + qy + cz = 0,\ ax + by + rz = 0$ has a non-trivial solution and $p \neq a, q \neq b, r \neq c$, prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$.

Solution

Assume that the system $px + by + cz = 0, ax + qy + cz = 0, ax + by + rz = 0$ has a non-trivial solution.

So, we have $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$. Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ in the above equation,

we get

$$\begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0. \text{ That is, } \begin{vmatrix} p & b & c \\ -(p-a) & q-b & 0 \\ -(p-a) & 0 & r-c \end{vmatrix} = 0.$$

Since $p \neq a, q \neq b, r \neq c$, we get $(p-a)(q-b)(r-c)\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$.

So, we have $\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$.

Expanding the determinant, we get $\frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} = 0$.

That is, $\frac{p}{p-a} + \frac{q-(q-b)}{q-b} + \frac{r-(r-c)}{r-c} = 0 \Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$.

EXERCISE 1.7

1. Solve the following system of homogenous equations.
   (i) $3x + 2y + 7z = 0,\quad 4x - 3y - 2z = 0,\quad 5x + 9y + 23z = 0$
   (ii) $2x + 3y - z = 0,\quad x - y - 2z = 0,\quad 3x + y + 3z = 0$
2. Determine the values of λ for which the following system of equations $x + y + 3z = 0,\ 4x + 3y + \lambda z = 0,\ 2x + y + 2z = 0$ has
   (i) a unique solution (ii) a non-trivial solution.
3. By using Gaussian elimination method, balance the chemical reaction equation:
   $C_2H_6 + O_2 \rightarrow H_2O + CO_2$

EXERCISE 1.8

Choose the Correct or the most suitable answer from the given four alternatives:

1. If $|\text{adj}(\text{adj}\, A)| = |A|^9$, then the order of the square matrix A is

   (1) 3 (2) 4 (3) 2 (4) 5

2. If A is a 3×3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then $BB^T =$

   (1) A (2) B (3) I_3 (4) B^T

3. If $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \text{adj}\, A$ and $C = 3A$, then $\dfrac{|\text{adj}\, B|}{|C|} =$

   (1) $\dfrac{1}{3}$ (2) $\dfrac{1}{9}$ (3) $\dfrac{1}{4}$ (4) 1

4. If $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $A =$

   (1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ (3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

5. If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I_2 - A =$

   (1) A^{-1} (2) $\dfrac{A^{-1}}{2}$ (3) $3A^{-1}$ (4) $2A^{-1}$

6. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then $|\text{adj}\,(AB)| =$

   (1) -40 (2) -80 (3) -60 (4) -20

7. If $P = \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3×3 matrix A and $|A| = 4$, then x is

   (1) 15 (2) 12 (3) 14 (4) 11

8. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is

   (1) 0 (2) -2 (3) -3 (4) -1

9. If A, B and C are invertible matrices of some order, then which one of the following is not true?

   (1) $\text{adj}\, A = |A|\, A^{-1}$ (2) $\text{adj}(AB) = (\text{adj}\, A)(\text{adj}\, B)$

   (3) $\det A^{-1} = (\det A)^{-1}$ (4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

10. If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then $B^{-1} =$

    (1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ (2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

11. If $A^T A^{-1}$ is symmetric, then $A^2 =$

(1) A^{-1} (2) $(A^T)^2$ (3) A^T (4) $(A^{-1})^2$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1} =$

(1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$ (3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ (4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$

13. If $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is

(1) $\frac{-4}{5}$ (2) $\frac{-3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

14. If $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and $AB = I_2$, then $B =$

(1) $\left(\cos^2\frac{\theta}{2}\right)A$ (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$ (3) $(\cos^2\theta)I$ (4) $\left(\sin^2\frac{\theta}{2}\right)A$

15. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A(\text{adj } A) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then $k =$

(1) 0 (2) $\sin\theta$ (3) $\cos\theta$ (4) 1

16. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $\lambda A^{-1} = A$, then λ is

(1) 17 (2) 14 (3) 19 (4) 21

17. If $\text{adj } A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and $\text{adj } B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then $\text{adj}(AB)$ is

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$ (3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is

(1) 1 (2) 2 (3) 4 (4) 3

19. If $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively,

(1) $e^{(\Delta_2/\Delta_1)}, e^{(\Delta_3/\Delta_1)}$ (2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$ (4)) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then $\operatorname{adj}(\lambda A) = \lambda^n \operatorname{adj}(A)$.

(iv) $A(\operatorname{adj}A) = (\operatorname{adj}A)A = |A|I$

(1) Only (i) (2) (ii) and (iii) (3) (iii) and (iv) (4) (i), (ii) and (iv)

21. If $\rho(A) = \rho([A\,|\,B])$, then the system $AX = B$ of linear equations is

(1) consistent and has a unique solution (2) consistent

(3) consistent and has infinitely many solution (4) inconsistent

22. If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0, (\cos\theta)x - y + z = 0,$ $(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{4}$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda - 7 & \mu + 5 \end{bmatrix}$. The system has infinitely many solutions if

(1) $\lambda = 7, \mu \ne -5$ (2) $\lambda = -7, \mu = 5$ (3) $\lambda \ne 7, \mu \ne -5$ (4) $\lambda = 7, \mu = -5$

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is

(1) 2 (2) 4 (3) 3 (4) 1

25. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then $\operatorname{adj}(\operatorname{adj} A)$ is

(1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

SUMMARY

(1) Adjoint of a square matrix A = Transpose of the cofactor matrix of A.

(2) $A(\operatorname{adj} A) = (\operatorname{adj} A)A = |A| I_n$.

(3) $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$.

(4) (i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$ (iii) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$, where λ is a non-zero scalar.

(5) (i) $(AB)^{-1} = B^{-1}A^{-1}$. (ii) $(A^{-1})^{-1} = A$

(6) If A is a non-singular square matrix of order n, then

(i) $(\text{adj } A)^{-1} = \text{adj}\left(A^{-1}\right) = \frac{1}{|A|}A$ (ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj }(\text{adj } A) = |A|^{n-2}A$ (iv) $\text{adj}(\lambda A) = \lambda^{n-1}\text{adj}(A)$, λ is a nonzero scalar

(v) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$ (vi) $(\text{adj } A)^T = \text{adj}\left(A^T\right)$

(vii) $\text{adj}(AB) = (\text{adj} B)(\text{adj} A)$

(7) (i) $A^{-1} = \pm\frac{1}{\sqrt{|\text{adj } A|}}\text{adj } A$. (ii) $A = \pm\frac{1}{\sqrt{|\text{adj } A|}}\text{adj }(\text{adj } A)$.

(8) (i) A matrix A is orthogonal if $AA^T = A^T A = I$

(ii) A matrix A is orthogonal if and only if A is non-singular and $A^{-1} = A^T$

(9) Methods to solve the system of linear equations $AX = B$

(i) By matrix inversion method $X = A^{-1}B, |A| \neq 0$

(ii) By Cramer's rule $x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}, \Delta \neq 0$.

(iii) By Gaussian elimination method

(10) (i) If $\rho(A) = \rho([A\,|\,B]) =$ number of unknowns, then the system has unique solution.

(ii) If $\rho(A) = \rho([A\,|\,B]) <$ number of unknowns, then the system has infinitely many solutions.

(iii) If $\rho(A) \neq \rho([A\,|\,B])$ then the system is inconsistent and has no solution.

(11) The homogenous system of linear equations $AX = 0$

(i) has the trivial solution, if $|A| \neq 0$.

(ii) has a non trivial solution, if $|A| = 0$.

ICT CORNER

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Open the Browser, type the URL Link given below (or) Scan the QR code. GeoGebra work book named "12th Standard Mathematics" will open. In the left side of the work book there are many chapters related to your text book. Click on the chapter named "Applications of Matrices and Determinants". You can see several work sheets related to the chapter. Select the work sheet "Application Matrices–Problem"

Chapter 2 Complex Numbers

"Imaginary numbers are a fine and wonderful refuge of the divine spirit almost an amphibian between being and non-being."

- Gottfried Leibniz

Rafael Bombelli (1526-1572)

Many mathematicians contributed to the full development of complex numbers. The rules for addition, subtraction, multiplication, and division of complex numbers were developed by the Italian mathematician Rafael Bombelli. He is generally regarded as the first person to develop an algebra of complex numbers. In honour of his accomplishments, a moon crater was named Bombelli.

Real Life Context

Complex numbers are useful in representing a phenomenon that has two parts varying at the same time, for instance an alternating current. Engineers, doctors, scientists, vehicle designers and others who use electromagnetic signals need to use complex numbers for strong signal to reach its destination. Complex numbers have essential concrete applications in signal processing, control theory, electromagnetism, fluid dynamics, quantum mechanics, cartography, and vibration analysis.

Learning Objectives

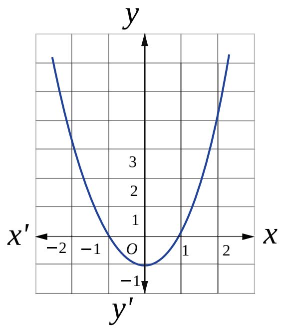
Upon completion of this chapter, students will be able to:

- perform algebraic operations on complex numbers
- plot the complex numbers in Argand plane
- find the conjugate and modulus of a complex number
- find the polar form and Euler form of a complex number
- apply de Moivre theorem to find the n^{th} roots of complex numbers.

2.1 Introduction to Complex Numbers

Before introducing complex numbers, let us try to answer the question "Whether there exists a real number whose square is negative?" Let's look at simple examples to get the answer for it. Consider the equations 1 and 2.

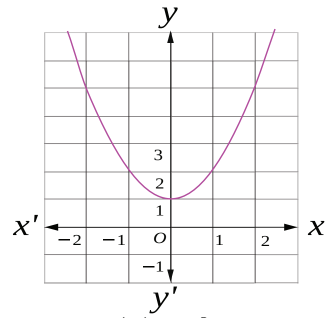
Equation 1	Equation 2
$x^2 - 1 = 0$ $x = \pm\sqrt{1}$ $x = \pm 1$	$x^2 + 1 = 0$ $x = \pm\sqrt{-1}$ $x = \pm ?$



$f(x) = x^2 - 1$

Fig. 2.1

Equation 1 has two real solutions, $x = -1$ and $x = 1$. We know that solving an equation in x is equivalent to finding the x-intercepts of a graph of $f(x) = x^2 - 1$ crosses the x-axis at $(-1, 0)$ and $(1, 0)$.



$f(x) = x^2 + 1$

Fig. 2.2

By the same logic, **equation 2** has no real solutions since the graph of $f(x) = x^2 + 1$ does not cross the x-axis; we can see this by looking at the graph of $f(x) = x^2 + 1$.

This is because, when we square a real number it is impossible to get a negative real number. If equation 2 has solutions, then we must create an imaginary number as a square root of -1. This imaginary unit $\sqrt{-1}$ is denoted by i. The imaginary number i tells us that $i^2 = -1$. We can use this fact to find other powers of i.

2.1.1 Powers of imaginary unit i

$i^0 = 1,\ i^1 = i$	$i^2 = -1$	$i^3 = i^2 i = -i$	$i^4 = i^2 i^2 = 1$
$(i)^{-1} = \frac{1}{i} = \frac{i}{(i)^2} = -i$	$(i)^{-2} = -1$	$(i)^{-3} = i$	$(i)^{-4} = 1 = i^4$

We note that, for any integer n, i^n has only four possible values: they correspond to values of n when divided by 4 leave the remainders 0, 1, 2, and 3. That is when the integer $n \le -4$ or $n \ge 4$, using division algorithm, n can be written as $n = 4q + k$, $0 \le k < 4$, k and q are integers and we write

$$(i)^n = (i)^{4q+k} = (i)^{4q}(i)^k = \left((i)^4\right)^q (i)^k = (1)^q (i)^k = (i)^k$$

Example 2.1

Simplify the following

(i) i^7 (ii) i^{1729} (iii) $i^{-1924} + i^{2018}$ (iv) $\sum_{n=1}^{102} i^n$ (v) $i\, i^2\, i^3 \cdots i^{40}$

Solution

(i) $(i)^7 = (i)^{4+3} = (i)^3 = -i$; (ii) $i^{1729} = i^{1728}\, i^1 = i$

(iii) $(i)^{-1924} + (i)^{2018} = (i)^{-1924+0} + (i)^{2016+2} = (i)^0 + (i)^2 = 1 - 1 = 0$

(iv) $\sum_{n=1}^{102} i^n = \left(i^1+i^2+i^3+i^4\right)+\left(i^5+i^6+i^7+i^8\right)+\cdots+\left(i^{97}+i^{98}+i^{99}+i^{100}\right)+i^{101}+i^{102}$

$= \left(i^1+i^2+i^3+i^4\right)+\left(i^1+i^2+i^3+i^4\right)+\cdots+\left(i^1+i^2+i^3+i^4\right)+i^1+i^2$

$= \{i+(-1)+(-i)+1\}+\{i+(-1)+(-i)+1\}+\cdots\ldots+\{i+(-1)+(-i)+1\}+i+(-1)$

$= 0+0+\cdots 0+i-1$

$= -1+i$ (What is this number?)

(v) $i\,i^2\,i^3\cdots i^{40} = i^{1+2+3+\cdots+40} = i^{\frac{40\text{x}41}{2}} = i^{820} = i^0 = 1.$

Result: Sum of four consecutive powers of i is zero. That is $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0 \;\; \forall n \in \mathbb{Z}$

Note

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ valid only if at least one of a, b is non-negative.

For example, $6 = \sqrt{36} = \sqrt{(-4)(-9)} = \sqrt{(-4)}\sqrt{(-9)} = (2i)(3i) = 6i^2 = -6$, a contradiction.

Since we have taken $\sqrt{(-4)(-9)} = \sqrt{(-4)}\sqrt{(-9)}$, we arrived at a contradiction.

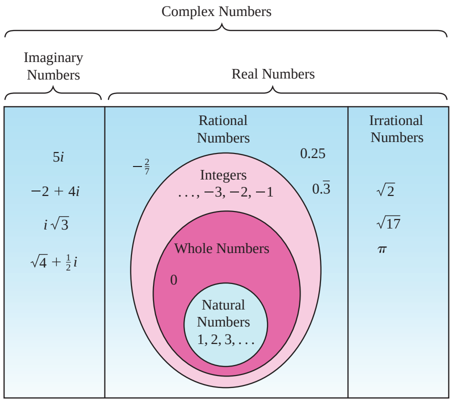
Therefore $\sqrt{ab} = \sqrt{a}\sqrt{b}$ valid only if at least one of a, b is non-negative.

(ii) For $y \in \mathbb{R}$, $y^2 \geq 0$

Therefore, $\sqrt{(-1)(y^2)} = \sqrt{(y^2)(-1)}$

$\sqrt{(-1)}\sqrt{(y^2)} = \sqrt{(y^2)}\sqrt{(-1)}$

$iy = yi.$



EXERCISE 2.1

Simplify the following:

1. $i^{1947} + i^{1950}$
2. $i^{1948} - i^{-1869}$
3. $\sum_{n=1}^{12} i^n$
4. $i^{59} + \frac{1}{i^{59}}$
5. $i\,i^2 i^3 \cdots i^{2000}$
6. $\sum_{n=1}^{10} i^{n+50}$

2.2 Complex Numbers

We have seen that the equation $x^2+1=0$ does not have a solution in real number system. In general there are polynomial equations with real coefficient which have no real solution. We enlarge the real number system so as to accommodate solutions of such polynomial equations. This has triggered the mathematicians to define complex number system.

In this section, we define

(i) Complex numbers in rectangular form

(ii) Argand plane

(iii) Algebraic operations on complex numbers

The complex number system is an extension of real number system with imaginary unit i.

The imaginary unit i with the property $i^2 = -1$, is combined with two real numbers x and y by the process of addition and multiplication, we obtain a complex number $x+iy$. The symbol '+' is treated as vector addition. It was introduced by Carl Friedrich Gauss (1777-1855).

2.2.1 Rectangular form

Definition 2.1 (Rectangular form of a complex number)

A complex number is of the form $x+iy$ **(or** $x+yi$**)**, where x and y are real numbers. x is called the real part and y is called the imaginary part of the complex number.

If $x=0$, the complex number is said to be purely imaginary. If $y=0$, the complex number is said to be real. Zero is the only number which is at once real and purely imaginary. It is customary to denote the standard rectangular form of a complex number $x+iy$ as z and we write $x=\text{Re}(z)$ and $y=\text{Im}(z)$. For instance, $\text{Re}(5-i7)=5$ and $\text{Im}(5-i7)=-7$.

The numbers of the form $\alpha+i\beta,\ \beta\neq 0$ are called imaginary (non real complex) numbers.

The equality of complex numbers is defined as follows.

Definition 2.2

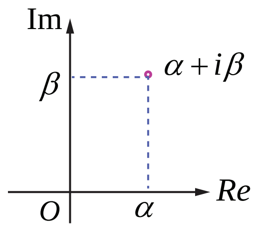
Two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are said to be equal if and only if **$\text{Re}(z_1)=\text{Re}(z_2)$ and $\text{Im}(z_1)=\text{Im}(z_2)$**. That is $x_1=x_2$ **and** $y_1=y_2$.

For instance, if $\alpha+i\beta=-7+3i$, then $\alpha=-7$ and $\beta=3$.

2.2.2 Argand plane

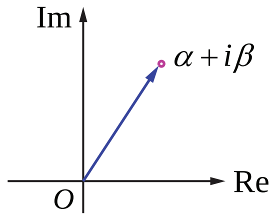
A complex number $z=x+iy$ is uniquely determined by an ordered pair of real numbers (x,y). The numbers $3-8i$, 6 and $-4i$ are equivalent to $(3,-8)$, $(6,0)$, and $(0,-4)$ respectively. In this way we are able to associate a complex number $z=x+iy$ with a point (x,y) in a coordinate plane. If we consider x axis as real axis and y axis as imaginary axis to represent a complex number, then the xy-plane is called complex plane or Argand plane. It is named after the Swiss mathematician Jean Argand (1768 – 1822).

A complex number is represented not only by a point, but also by a position vector pointing from the origin to the point. The number, the point, and the vector will all be denoted by the same letter z. As usual we identify all vectors which can be obtained from each other by parallel displacements. In this chapter, $\mathbb{C}$ denotes the set of all complex numbers. Geometrically, a complex number can be viewed as either a point in $\mathbb{R}^2$ or a vector in the Argand plane.



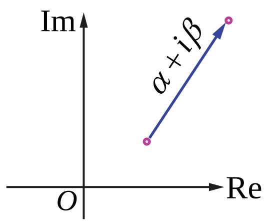
Complex number as a point

Fig. 2.3



Complex number by a position vector pointing from the origin to the point

Fig. 2.4

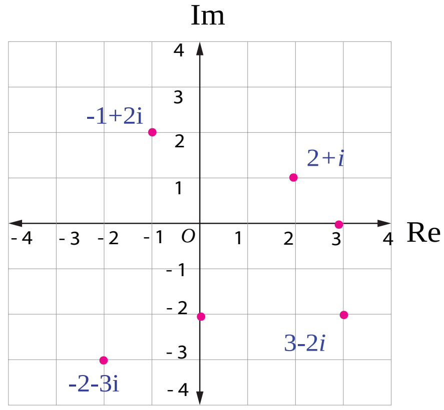


Complex number as a vector

Fig. 2.5

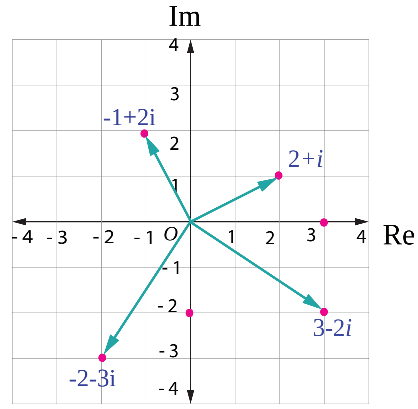
Illustration 2.1

Here are some complex numbers: $2+i,\ -1+2i,\ \ 3\text{-}2i,\ \ 0-2i,\ 3+\sqrt{-2},\ -2-3i,\ \cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$, and $3+0i$. Some of them are plotted in Argand plane.



Complex numbers as points

Fig. 2.6



Complex numbers as vectors

Fig. 2.7

2.2.3 Algebraic operations on complex numbers

In this section, we study the algebraic and geometric structure of the complex number system. We assume various corresponding properties of real numbers to be known.

(i) Scalar multiplication of complex numbers:

If $z = x + iy$ and $k \in \mathbb{R}$, then we define

$k\,z = (kx) + (ky)i$.

In particular $0z = 0$, $1z = z$ and $(-1)z = -z$.

The diagram below shows $k\,z$ for $k = 2, \frac{1}{2}, -1$

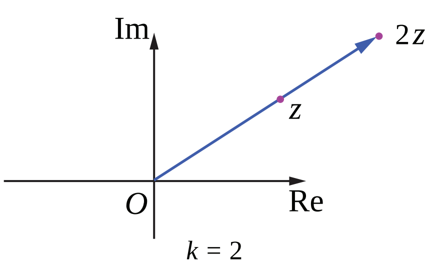


Fig. 2.8

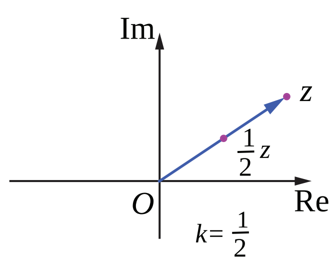


Fig. 2.9

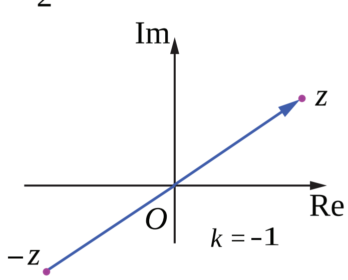


Fig. 2.10

(ii) Addition of complex numbers:

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $x_1, x_2, y_1,$ and $y_2 \in \mathbb{R}$, then we define

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2). \end{aligned}$$

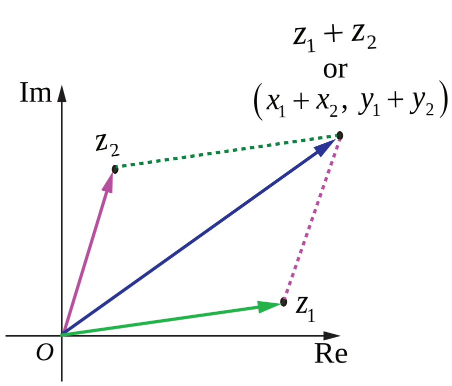


Fig. 2.11

We have already seen that vectors are characterized by length and direction, and that a given vector remains unchanged under translation. When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then by the parallelogram law of addition, the sum $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$ corresponds to the point $(x_1 + x_2,\ y_1 + y_2)$. It also corresponds to a vector with those coordinates as its components. Hence the points z_1, z_2, and $z_1 + z_2$ in complex plane may be obtained vectorially as shown in the adjacent Fig. 2.11.

(iii) Subtraction of complex numbers

Similarly the difference $z_1 - z_2$ can also be drawn as a position vector whose initial point is the origin and terminal point is $(x_1 - x_2,\ y_1 - y_2)$. We define

$$z_1 - z_2 = z_1 + (-z_2)$$
$$= (x_1 + iy_1) + (-x_2 - iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2).$$
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

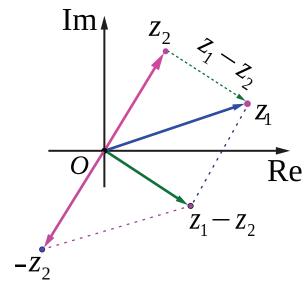


Fig. 2.12

It is important to note here that the vector representing the difference of the vector $z_1 - z_2$ may also be drawn joining the end point of z_2 to the tip of z_1 instead of the origin. This kind of representation does not alter the meaning or interpretation of the difference operator. The difference vector joining the tips of z_1 and z_2 is shown in (green) dotted line.

(iv) Multiplication of complex numbers

The multiplication of complex numbers z_1 and z_2 is defined as

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$
$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Although the product of two complex numbers z_1 and z_2 is itself a complex number represented by a vector, that vector lies in the same plane as the vectors z_1 and z_2. Evidently, then, this product is neither the scalar product nor the vector product used in vector algebra.

Remark

Multiplication of complex number *z* by *i*

If $z = x + iy$, then

$$iz = i(x + iy)$$
$$= -y + ix.$$

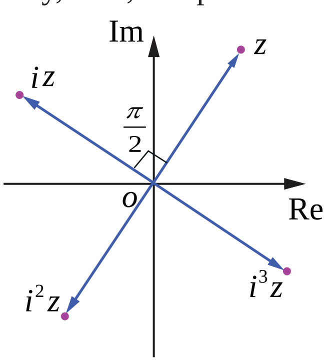


Fig. 2.13

The complex number iz is a rotation of z by 90° or $\frac{\pi}{2}$ radians in the counter clockwise direction about the origin. In general, multiplication of a complex number z by i successively gives a 90° counter clockwise rotation successively about the origin.

Illustration 2.2

Let $z_1 = 6 + 7i$ and $z_2 = 3 - 5i$. Then $z_1 + z_2$ and $z_1 - z_2$ are

(i) $(3 - 5i) + (6 + 7i) = (3 + 6) + (-5 + 7)i = 9 + 2i$

$(6 + 7i) - (3 - 5i) = (6 - 3) + (7 - (-5))i = 3 + 12i.$

Let $z_1 = 2 + 3i$ and $z_2 = 4 + 7i$. Then $z_1 z_2$ is

(ii) $(2 + 3i)(4 + 7i) = (2 \times 4 - 3 \times 7) + i(2 \times 7 + 3 \times 4)$

$= (8 - 21) + (14 + 12)i = -13 + 26i.$

Example 2.2

Find the value of the real numbers x and y, if the complex number $(2+i)x+(1-i)y+2i-3$ and $x+(-1+2i)y+1+i$ are equal

Solution

Let
$$z_1 = (2+i)x+(1-i)y+2i-3 = (2x+y-3)+i(x-y+2) \text{ and}$$
$$z_2 = x+(-1+2i)y+1+i = (x-y+1)+i(2y+1).$$

Given that $z_1 = z_2$.

Therefore $(2x+y-3)+i(x-y+2) = (x-y+1)+i(2y+1)$.

Equating real and imaginary parts separately, gives

$$2x+y-3 = x-y+1 \quad \Rightarrow \quad x+2y=4.$$
$$x-y+2 = 2y+1 \quad \Rightarrow \quad x-3y=-1.$$

Solving the above equations, gives

$$x=2 \text{ and } y=1.$$

EXERCISE 2.2

1. Evaluate the following if $z=5-2i$ and $w=-1+3i$

   (i) $z+w$ (ii) $z-iw$ (iii) $2z+3w$

   (iv) zw (v) $z^2+2zw+w^2$ (vi) $(z+w)^2$.

2. Given the complex number $z=2+3i$, represent the complex numbers in Argand diagram.

   (i) z, iz, and $z+iz$ (ii) z, $-iz$, and $z-iz$.

3. Find the values of the real numbers x and y, if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

2.3 Basic Algebraic Properties of Complex Numbers

The properties of addition and multiplication of complex numbers are the same as for real numbers. We list here the basic algebraic properties and verify some of them.

2.3.1 Properties of complex numbers

The complex numbers satisfy the following properties under **addition.**	The complex numbers satisfy the following properties under **multiplication.**
(i) **Closure property** For any two complex numbers z_1 and z_2, the sum z_1+z_2 is also a complex number.	(i) **Closure property** For any two complex numbers z_1 and z_2, the product $z_1 z_2$ is also a complex number.

(ii) **The commutative property** For any two complex numbers z_1 and z_2 $z_1 + z_2 = z_2 + z_1$.	(ii) **The commutative property** For any two complex numbers z_1 and z_2 $z_1 z_2 = z_2 z_1$.
(iii) **The associative property** For any three complex numbers z_1, z_2, and z_3 $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.	(iii) **The associative property** For any three complex numbers z_1, z_2, and z_3 $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
(iv) **The additive identity** There exists a complex number $0 = 0 + 0i$ such that, for every complex number z, $z + 0 = 0 + z = z$ The complex number $0 = 0 + 0i$ is known as additive identity.	(iv) **The multiplicative identity** There exists a complex number $1 = 1 + 0i$ such that, for every complex number z, $z1 = 1z = z$ The complex number $1 = 1 + 0i$ is known as multiplicative identity.
(v) **The additive inverse** For every complex number z there exists a complex number $-z$ such that, $z + (-z) = (-z) + z = 0.$ $-z$ is called the additive inverse of z.	(v) **The multiplicative inverse** For any nonzero complex number z, there exists a complex number w such that, $zw = wz = 1.$ w is called the multiplicative inverse of z. w is denoted by z^{-1}.
(vi) **Distributive property (multiplication distributes over addition)** For any three complex numbers z_1, z_2, and z_3 $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$.	

Let us now prove some of the properties.

Property

The commutative property under addition

For any two complex numbers z_1 and z_2, we have $z_1 + z_2 = z_2 + z_1$.

Proof

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and x_1, x_2, y_1, and $y_2 \in \mathbb{R}$,

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \\ &= (x_2 + x_1) + i(y_2 + y_1) \qquad (\text{since } x_1, x_2, y_1, \text{ and } y_2 \in \mathbb{R}) \\ &= (x_2 + iy_2) + (x_1 + iy_1) \\ &= z_2 + z_1. \end{aligned}$$

Property

Inverse Property under multiplication

Prove that the multiplicative inverse of a nonzero complex number $z = x + iy$ is $\dfrac{x}{x^2 + y^2} + i\dfrac{-y}{x^2 + y^2}$.

Proof

The multiplicative inverse is less obvious than the additive one.

Let $z^{-1} = u + iv$ be the inverse of $z = x + iy$

$$\begin{aligned} \text{We have } \quad z z^{-1} &= 1 \\ \text{That is } \quad (x+iy)(u+iv) &= 1 \\ (xu - yv) + i(xv + uy) &= 1 + i0 \end{aligned}$$

Equating real and imaginary parts we get

$$xu - yv = 1 \text{ and } xv + uy = 0.$$

Solving the above system of simultaneous equations in u and v

we get $u = \dfrac{x}{x^2+y^2}$ and $v = \dfrac{-y}{x^2+y^2}$. $\qquad$ ($\because$ z is non-zero $\Rightarrow x^2 + y^2 > 0$)

If $z = x + iy$, then $z^{-1} = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$. $\qquad$ ($\because$ z^{-1} is not defined when $z = 0$). ■

Note that the above example shows the existence of z^{-1} of the complex number z. To compute the inverse of a given complex number, we conveniently use $z^{-1} = \dfrac{1}{z}$. If z_1 and z_2 are two complex numbers where $z_2 \neq 0$, then the product of z_1 and $\dfrac{1}{z_2}$ is denoted by $\dfrac{z_1}{z_2}$. Other properties can be verified in a similar manner. In the next section, we define the conjugate of a complex number. It would help us to find the inverse of a complex number easily.

Complex numbers obey the laws of indices

(i) $z^m z^n = z^{m+n}$ $\qquad$ (ii) $\dfrac{z^m}{z^n} = z^{m-n}$, $z \neq 0$ $\qquad$ (iii) $\left(z^m\right)^n = z^{mn}$ $\qquad$ (iv) $\left(z_1 z_2\right)^m = z_1^m z_2^m$

EXERCISE 2.3

1. If $z_1 = 1 - 3i$, $z_2 = -4i$, and $z_3 = 5$, show that

   (i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ $\qquad$ (ii) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

2. If $z_1 = 3$, $z_2 = -7i$, and $z_3 = 5 + 4i$, show that

   (i) $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ $\qquad$ (ii) $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$.

3. If $z_1 = 2 + 5i$, $z_2 = -3 - 4i$, and $z_3 = 1 + i$, find the additive and multiplicative inverse of z_1, z_2, and z_3.

2.4 Conjugate of a Complex Number

In this section, we study about conjugate of a complex number, its geometric representation, and properties with suitable examples.

Definition 2.3

The conjugate of the complex number $x + iy$ is defined as the complex number $x - iy$.

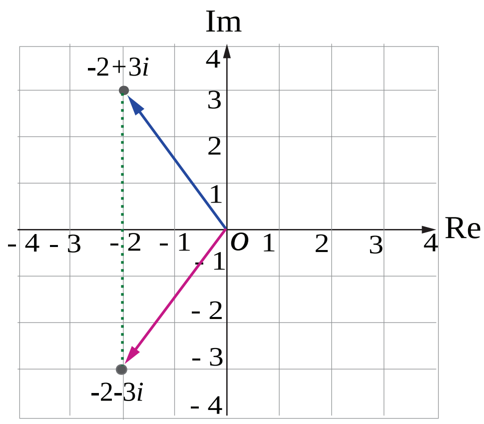
The complex conjugate of z is denoted by $\bar{z}$. To get the conjugate of the complex number z, simply change i by $-i$ in z. For instance $2-5i$ is the conjugate of $2+5i$. The product of a complex number with its conjugate is a real number.

For instance, (i) $(x+iy)(x-iy) = x^2-(iy)^2 = x^2+y^2$

(ii) $(1+3i)(1-3i) = (1)^2-(3i)^2 = 1+9 = 10$.

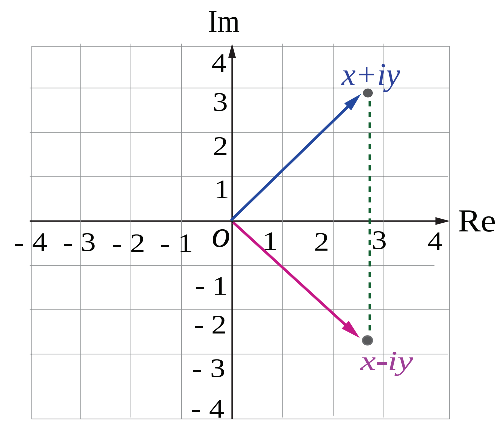
Geometrically, the conjugate of z is obtained by reflecting z on the real axis.

2.4.1 Geometrical representation of conjugate of a complex number



conjugate of a complex number

Fig. 2.14



conjugate of a complex number

Fig. 2.15

Note

Two complex numbers $x+iy$ and $x-iy$ are conjugates to each other. The conjugate is useful in division of complex numbers. The complex number can be replaced with a real number in the denominator by multiplying the numerator and denominator by the conjugate of the denominator. This process is similar to rationalising the denominator to remove surds.

2.4.2 Properties of Complex Conjugates

(1) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

(2) $\overline{z_1-z_2} = \bar{z}_1-\bar{z}_2$

(3) $\overline{z_1 z_2} = \bar{z}_1\,\bar{z}_2$

(4) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2},\; z_2 \neq 0$

(5) $\text{Re}(z) = \dfrac{z+\bar{z}}{2}$

(6) $\text{Im}(z) = \dfrac{z-\bar{z}}{2i}$

(7) $\overline{\left(z^n\right)} = \left(\bar{z}\right)^n$, where n is an integer

(8) z is real if and only if $z=\bar{z}$

(9) z is purely imaginary if and only if $z=-\bar{z}$

(10) $\bar{\bar{z}} = z$

Let us verify some of the properties.

Property

For any two complex numbers z_1 and z_2, we have $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$.

Proof

Let $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, and x_1, x_2, y_1, and $y_2 \in \mathbb{R}$

$$\overline{z_1+z_2} = \overline{(x_1+iy_1)+(x_2+iy_2)}$$

$$= \overline{(x_1+x_2)+i(y_1+y_2)} = (x_1+x_2)-i(y_1+y_2)$$

$$= (x_1-iy_1)+(x_2-iy_2)$$

$$= \overline{z_1}+\overline{z_2}.$$

It can be generalized by means of mathematical induction to sums involving any finite number of terms: $\overline{z_1+z_2+z_3+\cdots z_n} = \overline{z_1}+\overline{z_2}+\overline{z_3}+\cdots+\overline{z_n}$.

Property

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} \text{ where } x_1, x_2, y_1, \text{and } y_2 \in \mathbb{R}$$

Proof

Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$.

Then, $z_1z_2 = (x_1+iy_1)(x_2+iy_2) = (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$.

Therefore, $\overline{z_1z_2} = \overline{(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)} = (x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1)$,

and $\overline{z_1}\,\overline{z_2} = (x_1-iy_1)(x_2-iy_2) = (x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1)$.

Therefore, $\overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$.

Property

A complex number z is purely imaginary if and only if $z = -\overline{z}$

Proof

Let $z = x+iy$. Then by definition $\overline{z} = x-iy$

Therefore, $z = -\overline{z}$

$\Leftrightarrow \quad x+iy = -(x-iy)$

$\Leftrightarrow \quad 2x = 0 \Leftrightarrow x = 0$

$\Leftrightarrow$ z is purely imaginary.

Similarly, we can verify the other properties of conjugate of complex numbers.

Example 2.3

Write $\dfrac{3+4i}{5-12i}$ in the $x+iy$ form, hence find its real and imaginary parts.

Solution

To find the real and imaginary parts of $\dfrac{3+4i}{5-12i}$, first it should be expressed in the rectangular form $x+iy$. To simplify the quotient of two complex numbers, multiply the numerator and denominator by the conjugate of the denominator to eliminate i in the denominator.

$$\frac{3+4i}{5-12i} = \frac{(3+4i)(5+12i)}{(5-12i)(5+12i)}$$

$$= \frac{(15-48)+(20+36)i}{5^2+12^2}$$

$$= \frac{-33+56i}{169} = -\frac{33}{169}+i\frac{56}{169}.$$

Therefore, $\dfrac{3+4i}{5-12i} = -\dfrac{33}{169}+i\dfrac{56}{169}$. This is in the $x+iy$ form.

Hence real part is $-\dfrac{33}{169}$ and imaginary part is $\dfrac{56}{169}$.

Example 2.4

Simplify $\left(\dfrac{1+i}{1-i}\right)^3-\left(\dfrac{1-i}{1+i}\right)^3$. into rectangular form

Solution

We consider $\dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)}=\dfrac{1+2i-1}{1+1}=\dfrac{2i}{2}=i$,

and $\dfrac{1-i}{1+i} = \left(\dfrac{1+i}{1-i}\right)^{-1}=\dfrac{1}{i}=-i$.

Therefore, $\left(\dfrac{1+i}{1-i}\right)^3-\left(\dfrac{1-i}{1-i}\right)^3 = i^3-(-i)^3=-i-i=-2i$.

Example 2.5

If $\dfrac{z+3}{z-5i}=\dfrac{1+4i}{2}$, find the complex number z in the rectangular form

Solution

$$\text{We have } \frac{z+3}{z-5i}=\frac{1+4i}{2}$$

$$\Rightarrow 2(z+3) = (1+4i)(z-5i)$$

$$\Rightarrow 2z+6 = (1+4i)z+20-5i$$

$$\Rightarrow (2-1-4i)z = 20-5i-6$$

$$\Rightarrow z = \frac{14-5i}{1-4i}=\frac{(14-5i)(1+4i)}{(1-4i)(1+4i)}=\frac{34+51i}{17}=2+3i.$$

Example 2.6

If $z_1=3-2i$ and $z_2=6+4i$, find $\dfrac{z_1}{z_2}$ in the rectangular form

Solution

Using the given value for z_1 and z_2 the value of $\dfrac{z_1}{z_2} = \dfrac{3-2i}{6+4i}=\dfrac{3-2i}{6+4i}\times\dfrac{6-4i}{6-4i}$

$$= \frac{(18-8)+i(-12-12)}{6^2+4^2}=\frac{10-24i}{52}=\frac{10}{52}-\frac{24i}{52}$$

$$= \frac{5}{26}-\frac{6}{13}i.$$

Example 2.7

Find z^{-1}, if $z=(2+3i)(1-i)$.

Solution

$$\text{We have} \quad z = (2+3i)(1-i)=(2+3)+(3-2)i=5+i$$

$$\Rightarrow \quad z^{-1} = \frac{1}{z}=\frac{1}{5+i}.$$

Multiplying the numerator and denominator by the conjugate of the denominator, we get

$$z^{-1} = \frac{(5-i)}{(5+i)(5-i)}=\frac{5-i}{5^2+1^2}=\frac{5}{26}-i\frac{1}{26}$$

$$\Rightarrow \quad z^{-1} = \frac{5}{26}-i\frac{1}{26}.$$

Example 2.8

Show that (i) $\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}$ is real and (ii) $\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

Solution

(i) Let $z = \left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}$. Then, we get

$$\overline{z} = \overline{\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}}$$

$$= \overline{\left(2+i\sqrt{3}\right)^{10}}+\overline{\left(2-i\sqrt{3}\right)^{10}} \qquad \left(\because \overline{z_1+z_2}=\overline{z_1}+\overline{z_2}\right)$$

$$= \left(\overline{2+i\sqrt{3}}\right)^{10}+\left(\overline{2-i\sqrt{3}}\right)^{10} \qquad \left(\because \left(\overline{z^n}\right)=\left(\overline{z}\right)^n\right)$$

$$= \left(2-i\sqrt{3}\right)^{10}+\left(2+i\sqrt{3}\right)^{10}=z$$

$$\overline{z} = z \Rightarrow z \text{ is real.}$$

(ii) Let $z = \left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$.

$$\text{Here,} \quad \frac{19+9i}{5-3i} = \frac{(19+9i)(5+3i)}{(5-3i)(5+3i)}$$

$$= \frac{(95-27)+i(45+57)}{5^2+3^2}=\frac{68+102i}{34}$$

$$= 2+3i. \tag{1}$$

$$\text{and} \quad \frac{8+i}{1+2i} = \frac{(8+i)(1-2i)}{(1+2i)(1-2i)}$$

$$= \frac{(8+2)+i(1-16)}{1^2+2^2}=\frac{10-15i}{5}$$

$$= 2-3i. \tag{2}$$

Now $z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$

$\Rightarrow$ $z = (2+3i)^{15} - (2-3i)^{15}$. (by (1) and (2))

Then by definition, $\bar{z} = \left(\overline{(2+3i)^{15} - (2-3i)^{15}}\right)$

$= \left(\overline{2+3i}\right)^{15} - \left(\overline{2-3i}\right)^{15}$ (using properties of conjugates)

$= (2-3i)^{15} - (2+3i)^{15} = -\left((2+3i)^{15} - (2-3i)^{15}\right)$

$\Rightarrow$ $\bar{z} = -z$.

Therefore, $z = \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary. ∎

EXERCISE 2.4

1. Write the following in the rectangular form:

   (i) $\overline{(5+9i)+(2-4i)}$ (ii) $\frac{10-5i}{6+2i}$ (iii) $\overline{3i}+\frac{1}{2-i}$

2. If $z = x+iy$, find the following in rectangular form.

   (i) $\text{Re}\left(\frac{1}{z}\right)$ (ii) $\text{Re}(i\bar{z})$ (iii) $\text{Im}(3z+4\bar{z}-4i)$

3. If $z_1 = 2-i$ and $z_2 = -4+3i$, find the inverse of $z_1 z_2$ and $\frac{z_1}{z_2}$.

4. The complex numbers u, v, and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$.

   If $v = 3-4i$ and $w = 4+3i$, find u in rectangular form.

5. Prove the following properties:

   (i) z is real if and only if $z = \bar{z}$ (ii) $\text{Re}(z) = \frac{z+\bar{z}}{2}$ and $\text{Im}(z) = \frac{z-\bar{z}}{2i}$

6. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$

   (i) real (ii) purely imaginary.

7. Show that (i) $\left(2+i\sqrt{3}\right)^{10} - \left(2-i\sqrt{3}\right)^{10}$ is purely imaginary

   (ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

2.5 Modulus of a Complex Number

Just as the absolute value of a real number measures the distance of that number from origin along the real number line, the modulus of a complex number measures the distance of that number from the origin in the complex plane. Observe that the length of the line from the origin along the radial line to $z = x + iy$ is simply the hypotenuse of a right triangle, with one side of length x and the other side of length y.

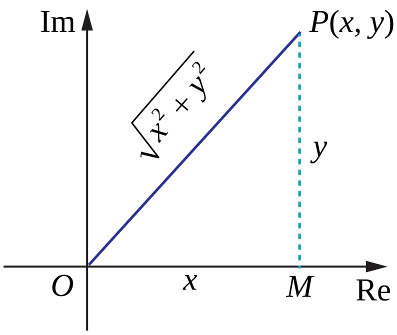


Fig. 2.16

Definition 2.4

If $z = x + iy$, then the modulus of z, denoted by $|z|$, is defined by $|z| = \sqrt{x^2 + y^2}$

For instance (i) $|i| = \sqrt{0^2 + 1^2} = 1$

(ii) $|-12i| = \sqrt{0^2 + (-12)^2} = 12$

(iii) $|12 - 5i| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$

Note

If $z = x + iy$, then $\bar{z} = x - iy$, then $z\bar{z} = (x + iy)(x - iy) = (x)^2 - (iy)^2 = x^2 + y^2 = |z|^2$.

$$\boxed{|z|^2 = z\bar{z}.}$$

2.5.1 Properties of Modulus of a complex number

(1) $|z| = |\bar{z}|$

(2) $|z_1 + z_2| \le |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1||z_2|$

(4) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

(5) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \ne 0$

(6) $|z^n| = |z|^n$, where n is an integer

(7) $\text{Re}(z) \le |z|$

(8) $\text{Im}(z) \le |z|$

Let us prove some of the properties.

Property Triangle inequality

For any two complex numbers z_1 and z_2, we have $|z_1 + z_2| \le |z_1| + |z_2|$.

Proof

Using
$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) \qquad (\because |z|^2 = z\bar{z})$$
$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \qquad (\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2)$$
$$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2$$
$$= z_1\bar{z}_1 + \left(z_1\bar{z}_2 + \overline{z_1\bar{z}_2}\right) + z_2\bar{z}_2 \qquad (\because \bar{\bar{z}} = z)$$

$$= |z_1|^2 + 2\,\text{Re}(z_1\bar{z}_2) + |z_2|^2 \qquad (\because\ 2\,\text{Re}(z) = z + \bar{z})$$

$$\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \qquad (\because\ \text{Re}(z) \leq |z|)$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \qquad (\because |z_1 z_2| = |z_1||z_2| \text{ and } |z| = |\bar{z}|)$$

$$\Rightarrow\ |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|.$$

Geometrical interpretation

Now consider the triangle shown in figure with vertices O, z_1 or z_2, and $z_1 + z_2$. We know from geometry that the length of the side of the triangle corresponding to the vector $z_1 + z_2$ cannot be greater than the sum of the lengths of the remaining two sides. This is the reason for calling the property as "Triangle Inequality".

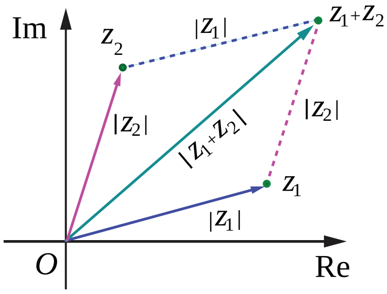


Fig. 2.17

It can be generalized by means of mathematical induction to finite number of terms:

$$|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n| \text{ for } n = 2, 3, \cdots.$$

Property The distance between the two points z_1 and z_2 in complex plane is $|z_1 - z_2|$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$|z_1 - z_2| = |(x_1 - x_2) + (y_1 - y_2)i|$$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Remark

The distance between the two points z_1 and z_2 in complex plane is $|z_1 - z_2|$.

If we consider origin, z_1 and z_2 as vertices of a triangle, by the similar argument we have

$$|z_1 - z_2| \leq |z_1| + |z_2|$$

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| \text{ and}$$

$$||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|.$$

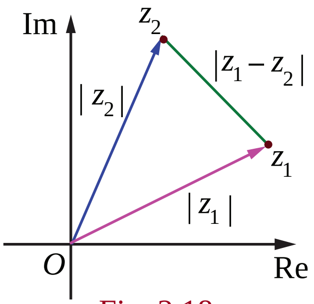


Fig. 2.18

Property Modulus of the product is equal to product of the moduli.

For any two complex numbers z_1 and z_2, we have $|z_1 z_2| = |z_1||z_2|$.

Proof

We have $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) \qquad (\because |z|^2 = z\bar{z})$

$$= (z_1)(z_2)(\overline{z_1})(\overline{z_2}) \qquad (\because\ \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2})$$

$$= \left(z_1 \overline{z_1}\right)\left(z_2 \overline{z_2}\right) = |z_1|^2 |z_2|^2 \qquad \text{(by commutativity } z_2 \overline{z_1} = \overline{z_1} z_2\text{)}$$

Therefore, $|z_1 z_2| = |z_1||z_2|$.

Note

It can be generalized by means of mathematical induction to any finite number of terms:

$$|z_1 z_2 z_3 \cdots z_n| = |z_1||z_2||z_3|\cdots|z_n|$$

That is the modulus value of a product of complex numbers is equal to the product of the moduli of complex numbers.

Similarly we can prove the other properties of modulus of a complex number.

Example 2.9

If $z_1 = 3+4i$, $z_2 = 5-12i$, and $z_3 = 6+8i$, find $|z_1|$, $|z_2|$, $|z_3|$, $|z_1+z_2|$, $|z_2-z_3|$, and $|z_1+z_3|$.

Solution

Using the given values for z_1, z_2 and z_3 we get $|z_1| = |3+4i| = \sqrt{3^2+4^2} = 5$

$$|z_2| = |5-12i| = \sqrt{5^2+(-12)^2} = 13$$

$$|z_3| = |6+8i| = \sqrt{6^2+8^2} = 10$$

$$|z_1+z_2| = |(3+4i)+(5-12i)| = |8-8i| = \sqrt{128} = 8\sqrt{2}$$

$$|z_2-z_3| = |(5-12i)-(6+8i)| = |-1-20i| = \sqrt{401}$$

$$|z_1+z_3| = |(3+4i)+(6+8i)| = |9+12i| = \sqrt{225} = 15$$

Note that the triangle inequality is satisfied in all the cases.

$|z_1+z_3| = |z_1|+|z_3| = 15$ (why?)

Example 2.10

Find the following (i) $\left|\dfrac{2+i}{-1+2i}\right|$ (ii) $\left|\overline{(1+i)}(2+3i)(4i-3)\right|$ (iii) $\left|\dfrac{i(2+i)^3}{(1+i)^2}\right|$

Solution

(i) $$\left|\frac{2+i}{-1+2i}\right| = \frac{|2+i|}{|-1+2i|} = \frac{\sqrt{2^2+1^2}}{\sqrt{(-1)^2+2^2}} = 1. \qquad \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0\right)$$

(ii) $$\left|\overline{(1+i)}(2+3i)(4i-3)\right| = \left|\overline{(1+i)}\right|\,|2+3i|\,|4i-3| \qquad \left(\because |z_1z_2z_3| = |z_1||z_2||z_3|\right)$$

$$= |1+i|\,|2+3i|\,|-3+4i| \qquad \left(\because |z| = |\overline{z}|\right)$$

$$= \left(\sqrt{1^2+1^2}\right)\left(\sqrt{2^2+3^2}\right)\left(\sqrt{(-3)^2+4^2}\right)$$

$$= \left(\sqrt{2}\right)\left(\sqrt{13}\right)\left(\sqrt{25}\right) = 5\sqrt{26}.$$

(iii) $$\left|\frac{i(2+i)^3}{(1+i)^2}\right| = \frac{|i|\,|(2+i)^3|}{|(1+i)^2|} = \frac{1|2+i|^3}{|1+i|^2} = \frac{\left(\sqrt{4+1}\right)^3}{\left(\sqrt{2}\right)^2} \qquad \left(\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0\right)$$

$$= \frac{\left(\sqrt{5}\right)^3}{2} = \frac{5\sqrt{5}}{2}.$$

Example 2.11

Which one of the points $i, -2+i$, and 3 is farthest from the origin?

Solution

The distance between origin to $z = i, -2+i,$ and 3 are

$$|z| = |i| = 1$$

$$|z| = |-2+i| = \sqrt{(-2)^2+1^2} = \sqrt{5}$$

$$|z| = |3| = 3$$

Since $1 < \sqrt{5} < 3$, the farthest point from the origin is 3.

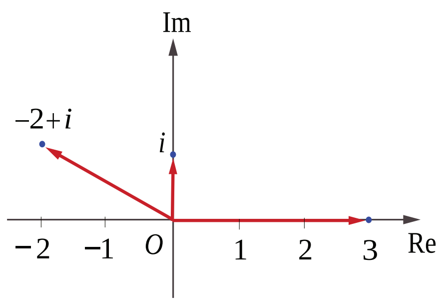


Fig. 2.19

Example 2.12

If z_1, z_2, and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = |z_1+z_2+z_3| = 1$, find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$.

Solution

Since, $|z_1| = |z_2| = |z_3| = 1,$

$$|z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1, |z_2|^2 = 1 \Rightarrow z_2\bar{z}_2 = 1, \text{ and } |z_3|^3 = 1 \Rightarrow z_3\bar{z}_3 = 1$$

Therefore, $\bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2}$, and $\bar{z}_3 = \frac{1}{z_3}$ and hence

$$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = |\bar{z}_1+\bar{z}_2+\bar{z}_3|$$

$$= |\overline{z_1+z_2+z_3}| = |z_1+z_2+z_3| = 1.$$

Example 2.13

If $|z| = 2$ show that $3 \le |z+3+4i| \le 7$

Solution

$$|z+3+4i| \le |z| + |3+4i| = 2+5 = 7$$

$$|z+3+4i| \le 7 \qquad (1)$$

$$|z+3+4i| \ge \big||z| - |3+4i|\big| = |2-5| = 3$$

$$|z+3+4i| \ge 3 \qquad (2)$$

From (1) and (2), we get $3 \le |z+3+4i| \le 7$.

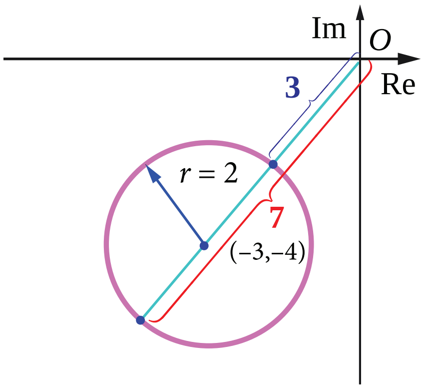


Fig. 2.20

Note

To find the lower bound and upper bound use $\big||z_1| - |z_2|\big| \le |z_1+z_2| \le |z_1| + |z_2|$.

Example 2.14

Show that the points $1,\ \dfrac{-1}{2}+i\dfrac{\sqrt{3}}{2},$ and $\dfrac{-1}{2}-i\dfrac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

Solution

It is enough to prove that the sides of the triangle are equal.

Let $z_1 = 1,\quad z_2 = \dfrac{-1}{2}+i\dfrac{\sqrt{3}}{2}$, and $z_3 = \dfrac{-1}{2}-i\dfrac{\sqrt{3}}{2}$.

The length of the sides of the triangles are

$$|z_1 - z_2| = \left|1-\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)\right| = \left|\frac{3}{2}-\frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{9}{4}+\frac{3}{4}} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$|z_2 - z_3| = \left|\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)-\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)\right| = \sqrt{\left(\sqrt{3}\right)^2} = \sqrt{3}$$

$$|z_3 - z_1| = \left|\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)-1\right| = \left|\frac{-3}{2}-\frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{9}{4}+\frac{3}{4}} = \sqrt{3}$$

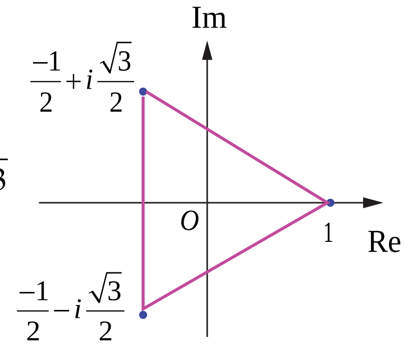


Fig. 2.21

Since the sides are equal, the given points form an equilateral triangle. ■

Example 2.15

Let $z_1, z_2,$ and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1 + z_2 + z_3 \neq 0$.

Prove that $\left|\dfrac{z_1z_2 + z_2z_3 + z_3z_1}{z_1+z_2+z_3}\right| = r$.

Solution

$$\text{Given that}\quad |z_1| = |z_2| = |z_3| = r \Rightarrow z_1\overline{z}_1 = z_2\overline{z}_2 = z_3\overline{z}_3 = r^2$$

$$\Rightarrow z_1 = \frac{r^2}{\overline{z}_1},\quad z_2 = \frac{r^2}{\overline{z}_2},\quad z_3 = \frac{r^2}{\overline{z}_3}$$

$$\text{Therefore}\quad z_1 + z_2 + z_3 = \frac{r^2}{\overline{z}_1} + \frac{r^2}{\overline{z}_2} + \frac{r^2}{\overline{z}_3}$$

$$= r^2\left(\frac{\overline{z}_2\overline{z}_3 + \overline{z}_1\overline{z}_3 + \overline{z}_1\overline{z}_2}{\overline{z}_1\overline{z}_2\overline{z}_3}\right)$$

$$|z_1 + z_2 + z_3| = |r^2|\left|\frac{\overline{z_2z_3 + z_1z_3 + z_1z_2}}{\overline{z_1z_2z_3}}\right| \qquad (\because \overline{z}_1 + \overline{z}_2 = \overline{z_1 + z_2})$$

$$= r^2\frac{|z_2z_3 + z_1z_3 + z_1z_2|}{|z_1||z_2||z_3|} \qquad (\because |z| = |\overline{z}| \text{ and } |z_1z_2z_3| = |z_1||z_2||z_3|)$$

$$|z_1 + z_2 + z_3| = r^2\frac{|z_2z_3 + z_1z_3 + z_1z_2|}{r^3} = \frac{|z_2z_3 + z_1z_3 + z_1z_2|}{r}$$

$$\Rightarrow \quad \frac{|z_2z_3 + z_1z_3 + z_1z_2|}{|z_1 + z_2 + z_3|} = r. \qquad \text{(given that } z_1 + z_2 + z_3 \neq 0\text{)}$$

Thus, $\left|\dfrac{z_2z_3 + z_1z_3 + z_1z_2}{z_1 + z_2 + z_3}\right| = r.$

Example 2.16

Show that the equation $z^2 = \overline{z}$ has four solutions.

Solution

We have, $z^2 = \overline{z}$.

$$\Rightarrow |z|^2 = |z|$$

$$\Rightarrow |z|(|z| - 1) = 0,$$

$$\Rightarrow |z| = 0, \text{ or } |z| = 1.$$

$$|z| = 0 \quad \Rightarrow z = 0 \text{ is a solution, } |z| = 1 \Rightarrow z\overline{z} = 1 \Rightarrow \overline{z} = \frac{1}{z}.$$

$$\text{Given } z^2 = \overline{z} \Rightarrow z^2 = \frac{1}{z} \Rightarrow z^3 = 1.$$

It has 3 non-zero solutions. Hence including zero solution, there are four solutions.

2.5.2 Square roots of a complex number

Let the square root of $a + ib$ be $x + iy$

$$\text{That is } \sqrt{a+ib} = x + iy \quad \text{where } x, y \in \mathbb{R}$$

$$a + ib = (x+iy)^2 = x^2 - y^2 + i2xy$$

Equating real and imaginary parts, we get

$$x^2 - y^2 = a \text{ and } 2xy = b$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = a^2 + b^2$$

$$x^2 + y^2 = \sqrt{a^2 + b^2}, \text{ since } x^2 + y^2 \text{ is positive}$$

Solving $x^2 - y^2 = a$ and $x^2 + y^2 = \sqrt{a^2+b^2}$, we get

$$x = \pm\sqrt{\frac{\sqrt{a^2+b^2} + a}{2}}\ ; \ y = \pm\sqrt{\frac{\sqrt{a^2+b^2} - a}{2}}.$$

Since $2xy = b$ it is clear that both x and y will have the same sign when b is positive, and x and y have different signs when b is negative.

$$\text{Therefore } \sqrt{a+ib} = \pm\left(\sqrt{\frac{|z| + a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z| - a}{2}}\right), \text{ where } b \neq 0. \quad (\because \operatorname{Re}(z) \leq |z|)$$

Formula for finding square root of a complex number

$$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z| + a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z| - a}{2}}\right), \text{ where } z = a + ib \text{ and } b \neq 0.$$

Note

If b is negative, $\frac{b}{|b|} = -1$, x and y have different signs.

If b is positive, $\frac{b}{|b|} = 1$, x and y have same sign.

Example 2.17

Find the square root of $6-8i$.

Solution

We compute $|6-8i| = \sqrt{6^2+(-8)^2} = 10$

and applying the formula for square root, we get

$$\begin{aligned}\sqrt{6-8i} &= \pm\left(\sqrt{\frac{10+6}{2}} - i\sqrt{\frac{10-6}{2}}\right) \qquad \left(\because b \text{ is negative, } \frac{b}{|b|} = -1\right)\\ &= \pm\left(\sqrt{8} - i\sqrt{2}\right)\\ &= \pm\left(2\sqrt{2} - i\sqrt{2}\right).\end{aligned}$$

EXERCISE 2.5

1. Find the modulus of the following complex numbers
   (i) $\frac{2i}{3+4i}$ (ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$ (iii) $(1-i)^{10}$ (iv) $2i(3-4i)(4-3i)$.
2. For any two complex numbers z_1 and z_2, such that $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$, then show that $\frac{z_1+z_2}{1+z_1 z_2}$ is a real number.
3. Which one of the points $10-8i$, $11+6i$ is closest to $1+i$.
4. If $|z| = 3$, show that $7 \leq |z+6-8i| \leq 13$.
5. If $|z| = 1$, show that $2 \leq |z^2-3| \leq 4$.
6. If $|z| = 2$, show that $8 \leq |z+6+8i| \leq 12$.
7. If z_1, z_2, and z_3 are three complex numbers such that $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1+z_2+z_3| = 1$, show that $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$.
8. If the area of the triangle formed by the vertices z, iz, and $z+iz$ is 50 square units, find the value of $|z|$.
9. Show that the equation $z^3 + 2\overline{z} = 0$ has five solutions.
10. Find the square roots of (i) $4+3i$ (ii) $-6+8i$ (iii) $-5-12i$.

2.6 Geometry and Locus of Complex Numbers

In this section let us study the geometrical interpretation of complex number z in complex plane and the locus of z in Cartesian form.

Example 2.18

Given the complex number $z = 3 + 2i$, represent the complex numbers z, iz, and $z + iz$ in one Argand plane. Show that these complex numbers form the vertices of an isosceles right triangle.

Solution

Given that $z = 3 + 2i$.

Therefore, $iz = i(3 + 2i) = -2 + 3i$

$z + iz = (3 + 2i) + i(3 + 2i) = 1 + 5i$

Let A, B, and C be z, $z + iz$, and iz respectively.

$AB^2 = |(z + iz) - z|^2 = |-2 + 3i|^2 = 13$

$BC^2 = |iz - (z + iz)|^2 = |-3 - 2i|^2 = 13$

$CA^2 = |z - iz|^2 = |5 - i|^2 = 26$

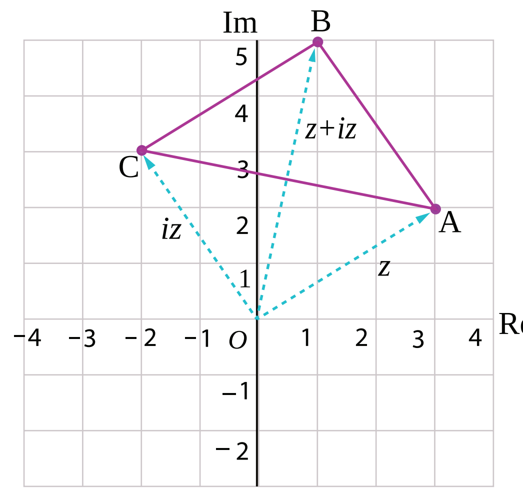


Fig. 2.22

Since $AB^2 + BC^2 = CA^2$ and $AB = BC$, ΔABC is an isosceles right triangle. ∎

Definition 2.5 (circle)

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always a constant. The fixed point is the centre and the constant distant is the radius of the circle.

Equation of Complex Form of a Circle

The locus of z that satisfies the equation $|z - z_0| = r$ where z_0 is a fixed complex number and r is a fixed positive real number consists of all points z whose distance from z_0 is r.

Therefore $|z - z_0| = r$ is the complex form of the equation of a circle. (see Fig. 2.23)

(i) $|z - z_0| < r$ represents the points interior of the circle.

(ii) $|z - z_0| > r$ represents the points exterior of the circle.

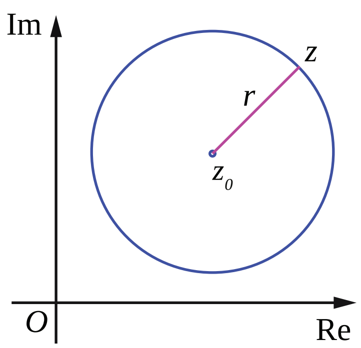


Fig. 2.23

Illustration 2.3

$|z| = r \Rightarrow \sqrt{x^2 + y^2} = r$

$\Rightarrow x^2 + y^2 = r^2$, represents a circle centre at the origin with radius r units.

Example 2.19

Show that $|3z-5+i|=4$ represents a circle, and, find its centre and radius.

Solution

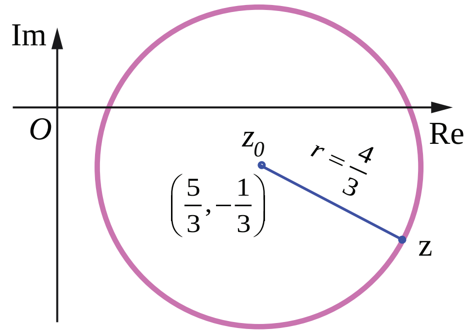


Fig. 2.24

The given equation $|3z-5+i|=4$ can be written as

$$3\left|z-\frac{5-i}{3}\right| = 4 \Rightarrow \left|z-\left(\frac{5}{3}-\frac{i}{3}\right)\right| = \frac{4}{3}.$$

It is of the form $|z-z_0|=r$ and so it represents a circle,

whose centre and radius are $\left(\frac{5}{3},-\frac{1}{3}\right)$ and $\frac{4}{3}$ respectively.

Example 2.20

Show that $|z+2-i|<2$ represents interior points of a circle. Find its centre and radius.

Solution

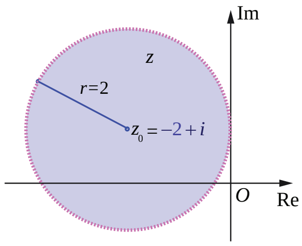


Fig. 2.25

Consider the equation $|z+2-i|=2$.

This can be written as $|z-(-2+i)|=2$.

The above equation represents the circle with centre $z_0=-2+i$ and radius $r=2$. Therefore $|z+2-i|<2$ represents all points inside the circle with centre at $-2+i$ and radius 2 as shown in figure.

Example 2.21

Obtain the Cartesian form of the locus of z in each of the following cases.

(i) $|z|=|z-i|$ (ii) $|2z-3-i|=3$

Solution

(i) we have $|z| = |z-i|$

$$\Rightarrow |x+iy| = |x+iy-i|$$

$$\Rightarrow \sqrt{x^2+y^2} = \sqrt{x^2+(y-1)^2}$$

$$\Rightarrow x^2+y^2 = x^2+y^2-2y+1$$

$$\Rightarrow 2y-1 = 0.$$

(ii) we have $|2z-3-i| = 3$

$$|2(x+iy)-3-i| = 3.$$

Squaring on both sides, we get

$$|(2x-3)+(2y-1)i|^2 = 9$$

$$\Rightarrow (2x-3)^2+(2y-1)^2 = 9$$

$\Rightarrow 4x^2+4y^2-12x-4y+1 = 0$, the locus of z in Cartesian form.

EXERCISE 2.6

1. If $z = x + iy$ is a complex number such that $\left|\dfrac{z-4i}{z+4i}\right| = 1$ show that the locus of z is real axis.

2. If $z = x + iy$ is a complex number such that $\operatorname{Im}\left(\dfrac{2z+1}{iz+1}\right) = 0$, show that the locus of z is $2x^2 + 2y^2 + x - 2y = 0$.

3. Obtain the Cartesian form of the locus of $z = x + iy$ in each of the following cases:

   (i) $\left[\operatorname{Re}(iz)\right]^2 = 3$ (ii) $\operatorname{Im}[(1-i)z+1] = 0$ (iii) $|z+i| = |z-1|$ (iv) $\bar{z} = z^{-1}$.

4. Show that the following equations represent a circle, and, find its centre and radius.

   (i) $|z-2-i| = 3$ (ii) $|2z+2-4i| = 2$ (iii) $|3z-6+12i| = 8$.

5. Obtain the Cartesian equation for the locus of $z = x + iy$ in each of the following cases:

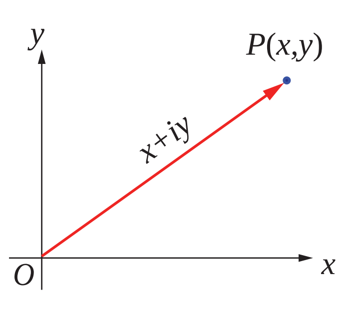
   (i) $|z-4| = 16$ (ii) $|z-4|^2 - |z-1|^2 = 16$.

2.7 Polar and Euler form of a Complex Number

When performing addition and subtraction of complex numbers, we use rectangular form. This is because we just add real parts and add imaginary parts; or subtract real parts, and subtract imaginary parts. When performing multiplication or finding powers or roots of complex numbers, use an alternate form namely, polar form, because it is easier to compute in polar form than in rectangular form.

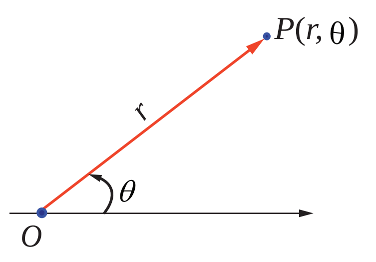
2.7.1 Polar form of a complex number

Polar coordinates form another set of parameters that characterize the vector from the origin to the point $z = x + iy$, with magnitude and direction. The polar coordinate system consists of a fixed point O called the pole and the horizontal half line emerging from the pole called the initial line (polar axis). If r is the distance from the pole to a point P and θ is an angle of inclination measured from the initial line in the counter clockwise direction to the line OP, then r and θ of the ordered pair (r, θ) are called the polar coordinates of P. Superimposing this polar coordinate system on the rectangular coordinate system, as shown in diagram, leads to



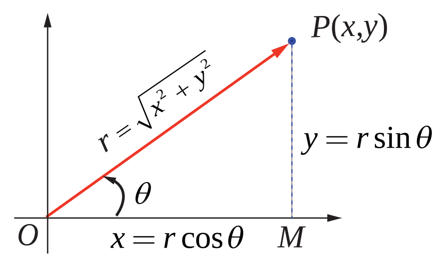
Rectangular coordinates

Fig. 2.26



Polar coordinates

Fig. 2.27



Superimpose polar coordinates on rectangular coordinates

Fig. 2.28

$$x = r\cos\theta \qquad \text{...(1)}$$

$$y = r\sin\theta. \qquad \text{...(2)}$$

Any non-zero complex number $z = x + iy$ can be expressed as $z = r\cos\theta + i\,r\sin\theta$.

Definition 2.6

Let r and θ be polar coordinates of the point $P(x, y)$ that corresponds to a non-zero complex number $z = x + iy$. The polar form or trigonometric form of a complex number P is

$$z = r(\cos\theta + i\sin\theta).$$

For convenience, we can write polar form as

$$z = x + iy = r\left(\cos\theta + i\sin\theta\right) = r\,cis\,\theta.$$

The value r represents the absolute value or modulus of the complex number z. The angle θ is called the argument or amplitude of the complex number z denoted by $\theta = \arg\left(z\right)$.

(i) If $z = 0$, the argument θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

(ii) If the complex number $z = x + iy$ has polar coordinates (r, θ), its conjugate $\bar{z} = x - iy$ has polar coordinates $(r, -\theta)$.

Squaring and adding (1) and (2), and taking square root, the value of r is given by $r = |z| = \sqrt{x^2 + y^2}$.

Dividing (2) by (1), $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{y}{x} \Rightarrow \tan\theta = \dfrac{y}{x}$.

Case (i) The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector. The angle θ has an infinitely many possible values, including negative ones that differ by integral multiples of 2π. Those values can be determined from the equation $\tan\theta = \dfrac{y}{x}$ where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an *argument* of z, and the set of all such values is obtained by adding multiple of 2π to θ, and it is denoted by arg z.

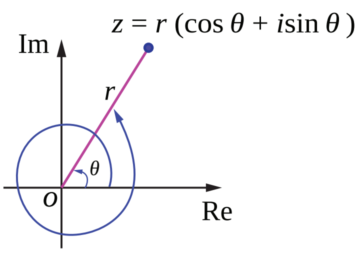


Fig. 2.29

Case (ii) There is a unique value of θ which satisfies the condition $-\pi < \theta \leq \pi$.

This value is called a principal value of θ or principal argument of z and is denoted by Arg z.

Note that, $$-\pi < \text{Arg}(z) \leq \pi \quad or \quad -\pi < \theta \leq \pi$$

Principal Argument of a complex number

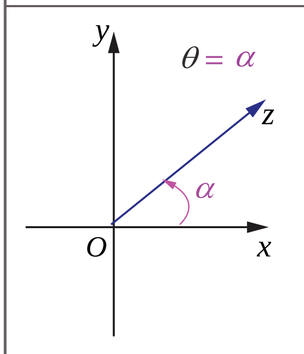
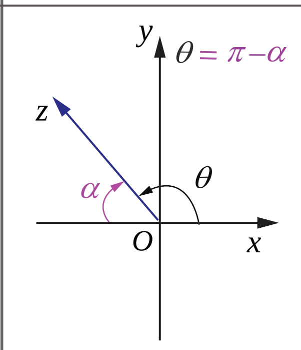
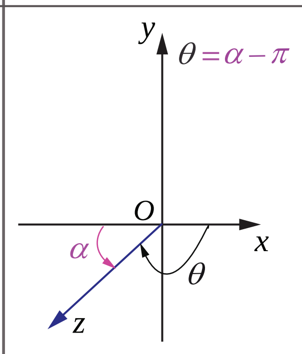
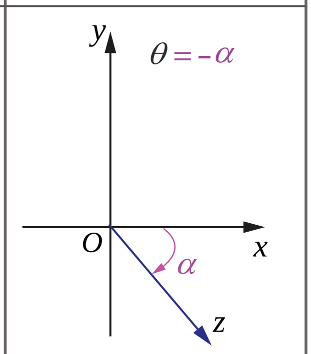
I-Quadrant	II-Quadrant	III-Quadrant	IV-Quadrant
			
$\theta = \alpha$	$\theta = \pi - \alpha$	$\theta = \alpha - \pi$	$\theta = -\alpha$

Fig. 2.30 Fig. 2.31 Fig. 2.32 Fig. 2.33

The capital A is important here to distinguish the principal value from the general value.

Evidently, in practice to find the principal angle θ, we usually compute $\alpha = \tan^{-1}\left|\frac{y}{x}\right|$ and adjust for the quadrant problem by adding or subtracting α with π appropriately.

$$\boxed{\arg z = Arg\, z + 2n\pi,\ n \in \mathbb{Z}.}$$

Some of the properties of arguments are

(1) $\arg\left(z_1\ z_2\right) = \arg z_1 + \arg z_2$

(2) $\arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

(3) $\arg\left(z^n\right) = n \arg z$

(4) The alternate forms of $\cos\theta + i\sin\theta$ are $\cos(2k\pi+\theta) + i\sin(2k\pi+\theta),\ k \in \mathbb{Z}$.

For instance the principal argument and argument of $1, i, -1$, and $-i$ are shown below:-

z	1	i	-1	$-i$
$Arg(z)$	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
$\arg z$	$2n\pi$	$2n\pi + \frac{\pi}{2}$	$2n\pi + \pi$	$2n\pi - \frac{\pi}{2}$

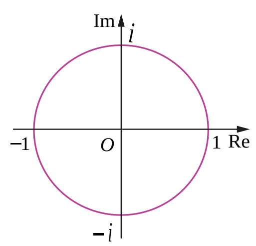


Fig. 2.34

Illustration

Plot the following complex numbers in complex plane

(i) $5\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$

(ii) $4\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

(iii) $3\left(\cos\dfrac{-5\pi}{6} + i\sin\dfrac{-5\pi}{6}\right)$

(iv) $2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)$.

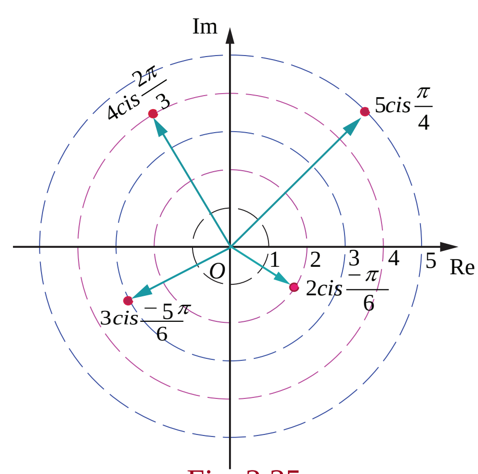


Fig. 2.35

2.7.2 Euler's Form of the complex number

The following identity is known as Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Euler formula gives the polar form $z = r\,e^{i\theta}$

Note

When performing multiplication or finding powers or roots of complex numbers, Euler form can also be used.

Example 2.22

Find the modulus and principal argument of the following complex numbers.

(i) $\sqrt{3}+i$ (ii) $-\sqrt{3}+i$ (iii) $-\sqrt{3}-i$ (iv) $\sqrt{3}-i$

Solution

(i) $\sqrt{3}+i$

$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{\left(\sqrt{3}\right)^2+1^2} = \sqrt{3+1} = 2$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

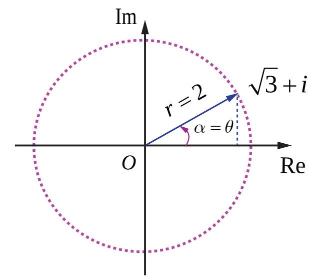


Fig. 2.36

Since the complex number $\sqrt{3}+i$ lies in the first quadrant, has the principal value

$$\theta = \alpha = \frac{\pi}{6}.$$

Therefore, the modulus and principal argument of $\sqrt{3}+i$ are 2 and $\frac{\pi}{6}$ respectively.

(ii) $-\sqrt{3}+i$

Modulus $= 2$ and

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

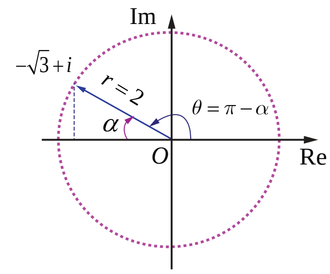


Fig. 2.37

Since the complex number $-\sqrt{3}+i$ lies in the second quadrant

has the principal value

$$\theta = \pi-\alpha = \pi-\frac{\pi}{6} = \frac{5\pi}{6}.$$

Therefore the modulus and principal argument of $-\sqrt{3}+i$ are 2 and $\frac{5\pi}{6}$ respectively.

(iii) $-\sqrt{3}-i$

$$r = 2 \text{ and } \alpha = \frac{\pi}{6}.$$

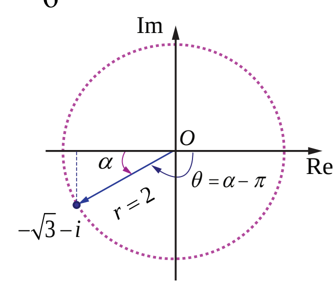


Fig. 2.38

Since the complex number $-\sqrt{3}-i$ lies in the third quadrant,

has the principal value,

$$\theta = \alpha-\pi = \frac{\pi}{6}-\pi = -\frac{5\pi}{6}.$$

Therefore, the modulus and principal argument of $-\sqrt{3}-i$ are 2 and $-\frac{5\pi}{6}$ respectively.

(iv) $\sqrt{3}-i$

$$r = 2 \text{ and } \alpha = \frac{\pi}{6}.$$

Since the complex number lies in the fourth quadrant, has the principal value,

$$\theta = -\alpha = -\frac{\pi}{6}$$

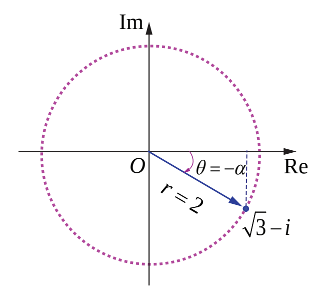


Fig. 2.39

Therefore, the modulus and principal argument of

$\sqrt{3}-i$ are 2 and $-\frac{\pi}{6}$.

In all the four cases, modulus are equal, but the arguments are depending on the quadrant in which the complex number lies. ■

Example 2.23

Represent the complex number (i) $-1-i$ (ii) $1+i\sqrt{3}$ in polar form.

Solution

(i) Let $-1-i = r(\cos\theta + i\sin\theta)$

$$\text{We have } r = \sqrt{x^2+y^2} = \sqrt{1^2+1^2} = \sqrt{1+1} = \sqrt{2}$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}1 = \frac{\pi}{4}.$$

Since the complex number $-1-i$ lies in the third quadrant, it has the principal value,

$$\theta = \alpha - \pi = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$

$$\text{Therefore, } -1-i = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right).$$

$$-1-i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}+2k\pi\right) - i\sin\left(\frac{3\pi}{4}+2k\pi\right)\right),\ k \in \mathbb{Z}.$$

Note

Depending upon the various values of k, we get various alternative polar forms.

(ii) $1+i\sqrt{3}$

$$r = |z| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$$

$$\text{Hence } \arg(z) = \frac{\pi}{3}.$$

Therefore, the polar form of $1+i\sqrt{3}$ can be written as

$$1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 2\left(\cos\left(\frac{\pi}{3}+2k\pi\right) + i\sin\left(\frac{\pi}{3}+2k\pi\right)\right),\ k \in \mathbb{Z}.$$

■

Example 2.24

Find the principal argument Arg z, when $z = \dfrac{-2}{1+i\sqrt{3}}$.

Solution

$$\begin{aligned}
\arg z &= \arg \frac{-2}{1+i\sqrt{3}} \\
&= \arg(-2) - \arg\left(1+i\sqrt{3}\right) \quad \left(\because \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\right) \\
&= \left(\pi - \tan^{-1}\left(\frac{0}{2}\right)\right) - \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \\
&= \pi - \frac{\pi}{3} = \frac{2\pi}{3}
\end{aligned}$$

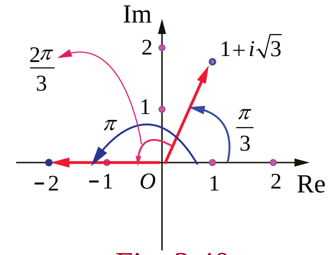


Fig. 2.40

This implies that one of the values of arg z is $\dfrac{2\pi}{3}$.

Since $\dfrac{2\pi}{3}$ lies between $-\pi$ and π, the principal argument Arg z is $\dfrac{2\pi}{3}$.

Properties of polar form

Property 1 If $z = r(\cos\theta + i\sin\theta)$, then $z^{-1} = \dfrac{1}{r}(\cos\theta - i\sin\theta)$.

Proof

$$\begin{aligned}
z^{-1} &= \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} \\
&= \frac{(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)} \\
&= \frac{(\cos\theta - i\sin\theta)}{r(\cos^2\theta + \sin^2\theta)} \\
z^{-1} &= \frac{1}{r}(\cos\theta - i\sin\theta).
\end{aligned}$$

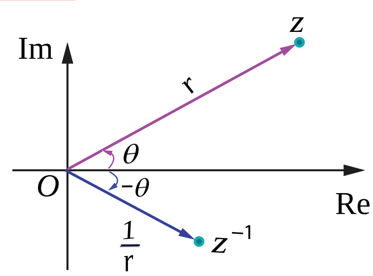


Fig. 2.41

Property 2

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $z_1 z_2 = r_1 r_2\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$.

Proof

$$\begin{aligned}
z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \text{ and} \\
z_2 &= r_2(\cos\theta_2 + i\sin\theta_2)
\end{aligned}$$

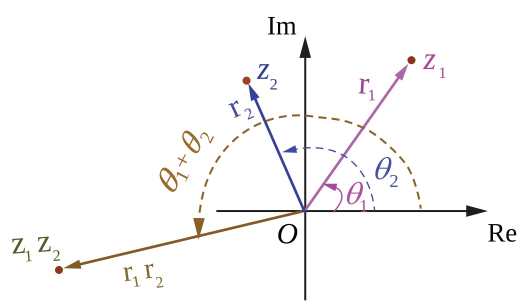


Fig. 2.42

$$\Rightarrow \; z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2\left(\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right)+i\left(\sin\theta_1\cos\theta_2+\sin\theta_2\cos\theta_1\right)\right)$$

$$z_1z_2 = r_1r_2\left(\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right).$$

Note

$$\arg\left(z_1z_2\right) = \theta_1+\theta_2 = \arg\left(z_1\right)+\arg\left(z_2\right).$$

Property 3

If $z_1 = r_1\left(\cos\theta_1+i\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2+i\sin\theta_2\right)$, then $\dfrac{z_1}{z_2}=\dfrac{r_1}{r_2}\left[\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right]$.

Proof: Using the polar form of z_1 and z_2, we have

$$\frac{z_1}{z_2} = \frac{r_1\left(\cos\theta_1+i\sin\theta_1\right)}{r_2\left(\cos\theta_2+i\sin\theta_2\right)}$$

$$= \frac{r_1\left(\cos\theta_1+i\sin\theta_1\right)\left(\cos\theta_2-i\sin\theta_2\right)}{r_2\left(\cos\theta_2+i\sin\theta_2\right)\left(\cos\theta_2-i\sin\theta_2\right)}$$

$$= \frac{r_1}{r_2}\,\frac{\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)+i\left(\sin\theta_1\cos\theta_2-\sin\theta_2\cos\theta_1\right)}{\cos^2\theta+\sin^2\theta}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right).$$

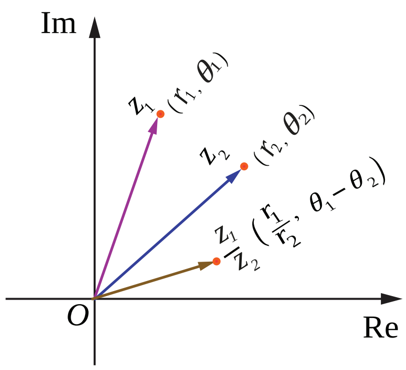


Fig. 2.43

Note

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1-\theta_2 = \arg\left(z_1\right)-\arg\left(z_2\right).$$

Example 2.25

Find the product $\frac{3}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\cdot 6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$ in rectangular from.

Solution:

The Product $\frac{3}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\cdot 6\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$$= \left(\frac{3}{2}\right)(6)\left(\cos\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{5\pi}{6}\right)\right)$$

$$= 9\left(\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right)$$

$$= 9\left(\cos\left(\pi+\frac{\pi}{6}\right)+i\sin\left(\pi+\frac{\pi}{6}\right)\right)$$

$$= 9\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right)$$

$$= 9\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = -\frac{9\sqrt{3}}{2} - \frac{9i}{2}$$, Which is in rectangular form.

Example 2.26

Find the quotient $\dfrac{2\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)\right)}$ in rectangular form.

Solution

$$\frac{2\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)\right)}$$

$$= \frac{1}{2}\left(\cos\left(\frac{9\pi}{4} - \left(\frac{-3\pi}{2}\right)\right) + i\sin\left(\frac{9\pi}{4} - \left(\frac{-3\pi}{2}\right)\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{9\pi}{4} + \frac{3\pi}{2}\right) + i\sin\left(\frac{9\pi}{4} + \frac{3\pi}{2}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{15\pi}{4}\right) + i\sin\left(\frac{15\pi}{4}\right)\right) = \frac{1}{2}\left(\cos\left(4\pi - \frac{\pi}{4}\right) + i\sin\left(4\pi - \frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$$

$$\frac{2\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right) + i\sin\left(\frac{-3\pi}{2}\right)\right)} = \frac{1}{2\sqrt{2}} - i\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} - i\frac{\sqrt{2}}{4}$$. Which is in rectangular form.

Example 2.27

If $z = x + iy$ and $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{2}$, show that $x^2 + y^2 = 1$.

Solution

Now, $$\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{\left[(x-1)+iy\right]\left[(x+1)-iy\right]}{\left[(x+1)+iy\right]\left[(x+1)-iy\right]}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{(x^2+y^2-1)+i(2y)}{(x+1)^2+y^2}.$$

Since, $$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2} \quad \Rightarrow \quad \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{2} \quad \Rightarrow \quad x^2+y^2-1 = 0$$

$$\Rightarrow \quad x^2+y^2 = 1.$$

EXERCISE 2.7

1. Write in polar form of the following complex numbers

   (i) $2+i2\sqrt{3}$ (ii) $3-i\sqrt{3}$ (iii) $-2-i2$ (iv) $\dfrac{i-1}{\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}}$.

2. Find the rectangular form of the complex numbers

   (i) $\left(\cos\dfrac{\pi}{6}+i\sin\dfrac{\pi}{6}\right)\left(\cos\dfrac{\pi}{12}+i\sin\dfrac{\pi}{12}\right)$ (ii) $\dfrac{\cos\dfrac{\pi}{6}-i\sin\dfrac{\pi}{6}}{2\left(\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}\right)}$.

3. If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n)=a+ib$, show that

   (i) $(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\cdots(x_n^2+y_n^2)=a^2+b^2$

   (ii) $\displaystyle\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=\tan^{-1}\left(\frac{b}{a}\right)+2k\pi,\ k\in\mathbb{Z}$.

4. If $\dfrac{1+z}{1-z}=\cos 2\theta+i\sin 2\theta$, show that $z=i\tan\theta$.

5. If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, show that

   (i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$ and

   (ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$.

6. If $z=x+iy$ and $\arg\left(\dfrac{z-i}{z+2}\right)=\dfrac{\pi}{4}$, show that $x^2+y^2+3x-3y+2=0$.

2.8 de Moivre's Theorem and its Applications

de Moivre
1667–1754

Abraham de Moivre (1667–1754) was one of the mathematicians to use complex numbers in trigonometry.

The formula $(\cos\theta+i\sin\theta)^n=(\cos n\theta+i\sin n\theta)$ known by his name, was instrumental in bringing trigonometry out of the realm of geometry and into that of analysis.

2.8.1 de Moivre's Theorem

de Moivre's Theorem

Given any complex number $\cos\theta+i\sin\theta$ and any integer n,

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

Corollary

(1) $(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$ (2) $(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta$

(3) $(\cos\theta-i\sin\theta)^{-n}=\cos n\theta+i\sin n\theta$ (4) $\sin\theta+i\cos\theta=i(\cos\theta-i\sin\theta)$.

Now let us apply de Moivre's theorem to simplify complex numbers and to find solution of equations.

Example 2.28

If $z=\left(\cos\theta+i\sin\theta\right)$, show that $z^{n}+\frac{1}{z^{n}}=2\cos n\theta$ and $z^{n}-\frac{1}{z^{n}}=2i\sin n\theta$.

Solution

Let $z=\left(\cos\theta+i\sin\theta\right)$.

By de Moivre's theorem ,

$$z^{n} = \left(\cos\theta+i\sin\theta\right)^{n}=\cos n\theta+i\sin n\theta$$

$$\frac{1}{z^{n}} = z^{-n}=\cos n\theta-i\sin n\theta$$

Therefore, $$z^{n}+\frac{1}{z^{n}} = \left(\cos n\theta+i\sin n\theta\right)+\left(\cos n\theta-i\sin n\theta\right)$$

$$z^{n}+\frac{1}{z^{n}} = 2\cos n\theta.$$

Similarly,

$$z^{n}-\frac{1}{z^{n}} = \left(\cos n\theta+i\sin n\theta\right)-\left(\cos n\theta-i\sin n\theta\right)$$

$$z^{n}-\frac{1}{z^{n}} = 2i\sin n\theta.$$

Example 2.29

Simplify $\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)^{18}$.

Solution

We have, $\sin\frac{\pi}{6}+i\cos\frac{\pi}{6} = i\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$.

Raising to the power 18 on both sides gives,

$$\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)^{18} = (i)^{18}\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^{18}$$

$$= (-1)\left(\cos\frac{18\pi}{6}-i\sin\frac{18\pi}{6}\right)$$

$$= -\left(\cos 3\pi-i\sin 3\pi\right)=1+0i.$$

Therefore, $$\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)^{18} = 1.$$

Example 2.30

Simplify $\left(\frac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)^{30}$.

Solution

Let $z = \cos 2\theta+i\sin 2\theta$.

As $|z| = |z|^{2}=z\bar{z}=1$, we get $\bar{z}=\frac{1}{z}=\cos 2\theta-i\sin 2\theta$.

Therefore, $\dfrac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta} = \dfrac{1+z}{1+\frac{1}{z}}=\dfrac{(1+z)z}{z+1}=z$.

Therefore, $\left(\dfrac{1+\cos 2\theta+i\sin 2\theta}{1+\cos 2\theta-i\sin 2\theta}\right)^{30} = z^{30}=\left(\cos 2\theta+i\sin 2\theta\right)^{30}$

$$= \cos 60\theta+i\sin 60\theta .$$

Example 2.31

Simplify (i) $(1+i)^{18}$ (ii) $(-\sqrt{3}+3i)^{31}$.

Solution

(i) $(1+i)^{18}$

Let $1+i = r(\cos\theta+i\sin\theta)$. Then, we get

$$r = \sqrt{1^2+1^2}=\sqrt{2}\ ;\ \alpha=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4},$$

$$\theta = \alpha=\frac{\pi}{4} \qquad (\because\ 1+i \text{ lies in the first Quadrant})$$

Therefore $1+i = \sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right)$

Raising to power 18 on both sides,

$$(1+i)^{18} = \left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{18}=\sqrt{2}^{18}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{18}.$$

By de Moivre's theorem,

$$(1+i)^{18} = 2^9\left(\cos\frac{18\pi}{4}+i\sin\frac{18\pi}{4}\right)$$

$$= 2^9\left(\cos\left(4\pi+\frac{\pi}{2}\right)+i\sin\left(4\pi+\frac{\pi}{2}\right)\right)=2^9\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$(1+i)^{18} = 2^9(i)=512\,i .$$

(ii) $(-\sqrt{3}+3i)^{31}$

Let $-\sqrt{3}+3i=r(\cos\theta+i\sin\theta)$. Then, we get

$$r = \sqrt{\left(-\sqrt{3}\right)^2+3^2}=\sqrt{12}=2\sqrt{3},$$

$$\alpha = \tan^{-1}\left|\frac{3}{-\sqrt{3}}\right|=\tan^{-1}\sqrt{3}=\frac{\pi}{3},$$

$$\theta = \pi-\alpha=\pi-\frac{\pi}{3}=\frac{2\pi}{3} \quad (\because\ -\sqrt{3}+3i \text{ lies in II Quadrant})$$

Therefore, $-\sqrt{3}+3i = 2\sqrt{3}\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$.

Raising power 31 on both sides,

$$
\begin{aligned}
(-\sqrt{3}+3i)^{31} &= \left(2\sqrt{3}\right)^{31}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{31}\\
&= \left(2\sqrt{3}\right)^{31}\left(\cos\left(20\pi+\frac{2\pi}{3}\right)+i\sin\left(20\pi+\frac{2\pi}{3}\right)\right)\\
&= \left(2\sqrt{3}\right)^{31}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\\
&= \left(2\sqrt{3}\right)^{31}\left(\cos\left(\pi-\frac{\pi}{3}\right)+i\sin\left(\pi-\frac{\pi}{3}\right)\right)\\
&= \left(2\sqrt{3}\right)^{31}\left(-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=\left(2\sqrt{3}\right)^{31}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right).
\end{aligned}
$$

2.8.2 Finding n^{th} roots of a complex number

de Moivre's formula can be used to obtain roots of complex numbers. Suppose n is a positive integer and a complex number ω is n^{th} root of z denoted by $z^{1/n}$, then we have

$$\omega^n = z. \qquad \text{...(1)}$$

Let $\omega = \rho\left(\cos\phi+i\sin\phi\right)$ and

$$z = r\left(\cos\theta+i\sin\theta\right)=r\left(\cos\left(\theta+2k\pi\right)+i\sin\left(\theta+2k\pi\right)\right),\ k\in\mathbb{Z}$$

Since w is the n^{th} root of z, then

$$\omega^n = z$$

$$\Rightarrow \quad \rho^n\left(\cos\phi+i\sin\phi\right)^n = r\left(\cos\left(\theta+2k\pi\right)+i\sin\left(\theta+2k\pi\right)\right),\ k\in\mathbb{Z}$$

By de Moivre's theorem,

$$\rho^n\left(\cos n\phi+i\sin n\phi\right) = r\left(\cos\left(\theta+2k\pi\right)+i\sin\left(\theta+2k\pi\right)\right),\ k\in\mathbb{Z}$$

Comparing the moduli and arguments, we get

$$\rho^n = r \text{ and } n\phi=\theta+2k\pi,\ k\in\mathbb{Z}$$

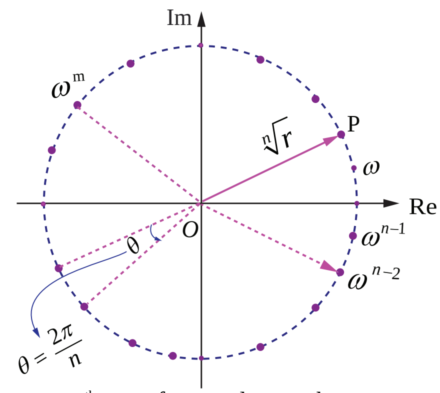
$$\rho = r^{1/n} \text{ and } \phi=\frac{\theta+2k\pi}{n},k\in\mathbb{Z}.$$

Therefore, the values of ω are $r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right),\ k\in\mathbb{Z}$.

Although there are infinitely many values of k, the distinct values of ω are obtained when $k=0,1,2,3,\ldots,n-1$. When $k=n,n+1,n+2,\ldots$ we get the same roots at regular intervals (cyclically). Therefore the n^{th} roots of complex number $z=r\left(\cos\theta+i\sin\theta\right)$ are

$$\boxed{z^{1/n}=r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right),\ k=0,1,2,3,\ldots,n-1.}$$

If we set $\omega = \sqrt[n]{r}e^{\frac{i(\theta+2k\pi)}{n}}$, the formula for the n^{th} roots of a complex number has a nice geometric interpretation, as shown in Figure. Note that because $|\omega| = \sqrt[n]{r}$ the n roots all have the same modulus $\sqrt[n]{r}$ they all lie on a circle of radius $\sqrt[n]{r}$ with centre at the origin. Furthermore, the n roots are equally spaced along the circle, because successive n roots have arguments that differ by $\frac{2\pi}{n}$.



n^{th} root of a complex number

Fig. 2.44

Remark

(1) General form of de Moivre's Theorem

If x is rational, then $\cos x\theta + i\sin x\theta$ is one of the values of $(\cos\theta + i\sin\theta)^x$.

(2) Polar form of unit circle

$$\text{Let } z = e^{i\theta} = \cos\theta + i\sin\theta. \text{ Then, we get}$$
$$|z|^2 = |\cos\theta + i\sin\theta|^2$$
$$\Rightarrow |x+iy|^2 = \cos^2\theta + \sin^2\theta = 1$$
$$\Rightarrow x^2 + y^2 = 1.$$

Therefore, $|z| = 1$ represents a unit circle (radius one) centre at the origin.

2.8.3 The n^{th} roots of unity

The solutions of the equation $z^n = 1$, for positive values of integer n, are the n roots of the unity. In polar form the equation $z^n = 1$ can be written as

$$z^n = \cos(0+2k\pi) + i\sin(0+2k\pi) = e^{i2k\pi}, \quad k = 0, 1, 2, \ldots.$$

Using deMoivre's theorem, we find the n^{th} roots of unity from the equation given below:

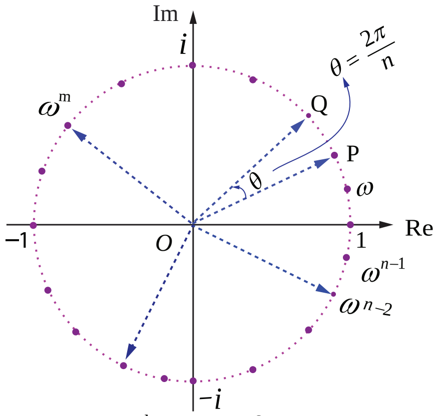
$$z = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right) = e^{\frac{i2k\pi}{n}}, k = 0,1,2,3,\ldots,n-1. \qquad \ldots (1)$$

Given a positive integer n, a complex number z is called an n^{th} root of unity if and only if $z^n = 1$.

If we denote the complex number by ω, then

$$\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi i}{n} + i\sin\frac{2\pi i}{n}$$
$$\Rightarrow \omega^n = \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i} = 1.$$

Therefore ω is an n^{th} root of unity. From equation (1), the complex numbers $1, \omega, \omega^2, \cdots, \omega^{n-1}$ are n^{th} roots of unity. The complex numbers $1, \omega, \omega^2, \cdots, \omega^{n-1}$ are the points in the complex plane and are the vertices of a regular polygon of n sides inscribed in a unit circle as shown in Fig 2.45. Note that because the n^{th} roots all have the same modulus 1, they will lie on a circle of radius 1 with centre at the origin. Furthermore, the n roots are equally spaced along the circle, because successive n^{th} roots have arguments that differ by $\dfrac{2\pi}{n}$.



n^{th} roots of unity

Fig. 2.45

The n^{th} roots of unity $1, \omega, \omega^2, \cdots, \omega^{n-1}$ are in geometric progression with common ratio ω.

Therefore $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = \dfrac{1-\omega^n}{1-\omega} = 0$ since $\omega^n = 1$ and $\omega \neq 1$.

The sum of all the n^{th} roots of unity is
$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$.

The product of n, n^{th} roots of unit is

$$1\omega\omega^2\cdots\omega^{n-1} = \omega^{0+1+2+3+\cdots+(n-1)} = \omega^{\frac{(n-1)n}{2}}$$

$$= \left(\omega^n\right)^{\frac{(n-1)}{2}} = \left(e^{i2\pi}\right)^{\frac{(n-1)}{2}} = \left(e^{i\pi}\right)^{n-1} = (-1)^{n-1}$$

The product of all the n^{th} roots of unity is
$1\omega\omega^2\cdots\omega^{n-1} = (-1)^{n-1}$.

Since $|\omega| = 1$, we have $\omega\bar{\omega} = |\omega|^2 = 1$; hence $\bar{\omega} = \omega^{-1} \Rightarrow (\bar{\omega})^k = \omega^{-k}$, $0 \le k \le n-1$

$$\omega^{n-k} = \omega^n\omega^{-k} = \omega^{-k} = (\bar{\omega})^k,\ 0 \le k \le n-1$$

Therefore, $\boxed{\omega^{n-k} = \omega^{-k} = \left(\bar{\omega}\right)^k,\ 0 \le k \le n-1.}$

Note

(1) All the n roots of n^{th} roots unity are in Geometrical Progression

(2) Sum of the n roots of n^{th} roots unity is always equal to zero.

(3) Product of the n roots of n^{th} roots unity is equal to $(-1)^{n-1}$.

(4) All the n roots of n^{th} roots unity lie on the circumference of a circle whose centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.

Example 2.32

Find the cube roots of unity.

Solution

We have to find $1^{\frac{1}{3}}$. Let $z = 1^{\frac{1}{3}}$ then $z^3 = 1$.

In polar form, the equation $z^3 = 1$ can be written as

$z^3 = \cos(0+2k\pi) + i\sin(0+2k\pi) = e^{i2k\pi}$, $k = 0, 1, 2, \cdots$.

Therefore, $z = \cos\left(\dfrac{2k\pi}{3}\right) + i\sin\left(\dfrac{2k\pi}{3}\right) = e^{i\frac{2k\pi}{3}}$, $k = 0, 1, 2$.

Taking $k = 0,1,2$, we get,

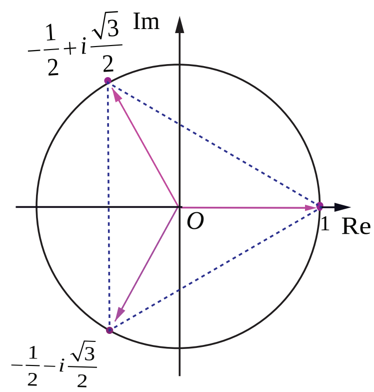
$$k=0, \quad z = \cos 0 + i\sin 0 = 1.$$

$$k=1, \quad z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$$

$$= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$k=2, \quad z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos\left(\pi + \frac{\pi}{3}\right) + i\sin\left(\pi + \frac{\pi}{3}\right)$$

$$= -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$



Cube roots of unity

Fig. 2.46

Therefore, the cube roots of unity are

$$1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2} \Rightarrow 1, \omega, \text{ and } \omega^2, \text{ where } \omega = e^{i\frac{2\pi}{3}} = \frac{-1+i\sqrt{3}}{2}.$$

Example 2.33

Find the fourth roots of unity.

Solution

We have to find $1^{\frac{1}{4}}$. Let $z = 1^{\frac{1}{4}}$. Then $z^4 = 1$.

In polar form, the equation $z^4 = 1$ can be written as

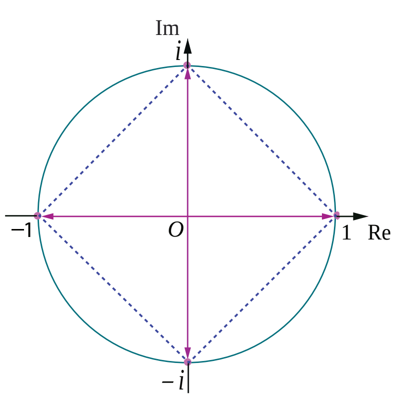
$z^4 = \cos(0+2k\pi) + i\sin(0+2k\pi) = e^{i2k\pi}, k = 0, 1, 2, \ldots$.

Therefore, $z = \cos\left(\dfrac{2k\pi}{4}\right) + i\sin\left(\dfrac{2k\pi}{4}\right) = e^{i\frac{2k\pi}{4}}$, $k = 0,1,2,3$.

Taking $k = 0,1,2,3$, we get

$$k=0, \quad z = \cos 0 + i\sin 0 = 1.$$

$$k=1, \quad z = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i.$$



Fourth roots of unity

Fig. 2.47

$k=2, \qquad z = \cos\pi + i\sin\pi = -1.$

$k=3, \qquad z = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -\cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i.$

Fourth roots of unity are $1, i, -1, -i \quad \Rightarrow \quad 1, \omega, \omega^2$, and ω^3, where $\omega = e^{i\frac{2\pi}{4}} = i$.

Note

(i) In this chapter the letter ω is used for n^{th} roots of unity. Therefore the value of ω is depending on n as shown in following table.

value of n	2	3	4	5	$\cdots$	k
value of ω	$e^{i\frac{2\pi}{2}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{4}}$	$e^{i\frac{2\pi}{5}}$	$\cdots$	$e^{i\frac{2\pi}{k}}$

(ii) The complex number $z e^{i\theta}$ is a rotation of z by θ radians in the counter clockwise direction about the origin.

Example 2.34

Solve the equation $z^3 + 8i = 0$, where $z \in \mathbb{C}$.

Solution

Let $\qquad z^3 + 8i = 0$. Then, we get

$$z^3 = -8i$$

$$= 8(-i) = 8\left(\cos\left(-\frac{\pi}{2}+2k\pi\right) + i\sin\left(-\frac{\pi}{2}+2k\pi\right)\right), k \in \mathbb{Z}.$$

Therefore, $z = \sqrt[3]{8}\left(\cos\left(\frac{-\pi+4k\pi}{6}\right) + i\sin\left(\frac{-\pi+4k\pi}{6}\right)\right),\ k = 0,1,2.$

Taking $k = 0,1,2$, we get,

$k=0, \qquad z = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i$

$k=1, \qquad z = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2 = 2(0+i) = 0+2i = 2i.$

$k=2, \qquad z = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) = 2\left(\cos\left(\pi+\frac{\pi}{6}\right) + i\sin\left(\pi+\frac{\pi}{6}\right)\right)$

$$= 2\left(-\cos\left(\frac{\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right) = 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i.$$

The values of z are $\sqrt{3}-i$, $2i$, and $-\sqrt{3}-i$.

Example 2.35

Find all cube roots of $\sqrt{3}+i$.

Solution

We have to find $\left(\sqrt{3}+i\right)^{\frac{1}{3}}$. Let $z=\left(\sqrt{3}+i\right)^{\frac{1}{3}}$. Then, $z^3 = \sqrt{3}+i = r\left(\cos\theta + i\sin\theta\right)$.

$$\text{Then,}\quad r = \sqrt{3+1}=2,\ \text{and}\ \alpha=\theta=\frac{\pi}{6} \qquad (\because \sqrt{3}+i \text{ lies in the first quadrant})$$

$$\text{Therefore,}\quad z^3 = \sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow\quad z = \sqrt[3]{2}\left(\cos\left(\frac{\pi+12k\pi}{18}\right)+i\sin\left(\frac{\pi+12k\pi}{18}\right)\right),\ k=0,1,2.$$

Taking $k=0,1,2$, we get

$$k=0,\qquad z = 2^{\frac{1}{3}}\left(\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right);$$

$$k=1,\qquad z = 2^{\frac{1}{3}}\left(\cos\frac{13\pi}{18}+i\sin\frac{13\pi}{18}\right);$$

$$k=2,\qquad z = 2^{\frac{1}{3}}\left(\cos\frac{25\pi}{18}+i\sin\frac{25\pi}{18}\right)=2^{\frac{1}{3}}\left(-\cos\frac{7\pi}{18}-i\sin\frac{7\pi}{18}\right).$$

Example 2.36

Suppose z_1, z_2, and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z|=2$. If $z_1=1+i\sqrt{3}$, then find z_2 and z_3.

Solution

$|z|=2$ represents the circle with centre $(0,0)$ and radius 2.

Let A, B, and C be the vertices of the given triangle. Since the vertices z_1, z_2, and z_3 form an equilateral triangle inscribed in the circle $|z|=2$, the sides of this triangle AB, BC, and CA subtend $\frac{2\pi}{3}$ radians (120 degree) at the origin (circumcenter of the triangle).

(The complex number $ze^{i\theta}$ is a rotation of z by θ radians in the counter clockwise direction about the origin.)

Therefore, we can obtain z_2 and z_3 by the rotation of z_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively.

Given that

$$\overrightarrow{OA} = z_1 = 1+i\sqrt{3};$$

$$\overrightarrow{OB} = z_1e^{i\frac{2\pi}{3}} = \left(1+i\sqrt{3}\right)e^{i\frac{2\pi}{3}}$$

$$= \left(1+i\sqrt{3}\right)\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$= \left(1+i\sqrt{3}\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-2;$$

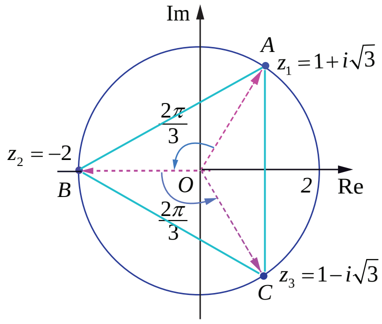


Fig. 2.48

$$\overrightarrow{OC} = z_1 e^{i\frac{4\pi}{3}} = z_2 e^{i\frac{2\pi}{3}} = -2e^{i\frac{2\pi}{3}}$$

$$= -2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= -2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}.$$

Therefore, $z_2 = -2$, and $z_3 = 1 - i\sqrt{3}$.

EXERCISE 2.8

1. If $\omega \neq 1$ is a cube root of unity, show that $\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \dfrac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1$.

2. Show that $\left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5 = -\sqrt{3}$.

3. Find the value of $\left(\dfrac{1+\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10} - i\cos\frac{\pi}{10}}\right)^{10}$.

4. If $2\cos\alpha = x + \dfrac{1}{x}$ and $2\cos\beta = y + \dfrac{1}{y}$, show that

   (i) $\dfrac{x}{y} + \dfrac{y}{x} = 2\cos(\alpha - \beta)$ (ii) $xy - \dfrac{1}{xy} = 2i\sin(\alpha + \beta)$

   (iii) $\dfrac{x^m}{y^n} - \dfrac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$ (iv) $x^m y^n + \dfrac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$.

5. Solve the equation $z^3 + 27 = 0$.

6. If $\omega \neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3 + 8 = 0$ are $-1,\ 1-2\omega,\ 1-2\omega^2$.

7. Find the value of $\displaystyle\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9} + i\sin\frac{2k\pi}{9}\right)$.

8. If $\omega \neq 1$ is a cube root of unity, show that

   (i) $(1-\omega+\omega^2)^6 + (1+\omega-\omega^2)^6 = 128$.

   (ii) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\cdots(1+\omega^{2^{11}}) = 1$.

9. If $z = 2-2i$, find the rotation of z by θ radians in the counter clockwise direction about the origin when

   (i) $\theta = \dfrac{\pi}{3}$ (ii) $\theta = \dfrac{2\pi}{3}$ (iii) $\theta = \dfrac{3\pi}{2}$.

EXERCISE 2.9

Choose the correct or the most suitable answer from the given four alternatives :

1. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

(1) 0 (2) 1 (3) -1 (4) i

2. The value of $\sum_{n=1}^{13} (i^n + i^{n-1})$ is

(1) $1+i$ (2) i (3) 1 (4) 0

3. The area of the triangle formed by the complex numbers $z, iz,$ and $z+iz$ in the Argand's diagram is

(1) $\frac{1}{2}|z|^2$ (2) $|z|^2$ (3) $\frac{3}{2}|z|^2$ (4) $2|z|^2$

4. The conjugate of a complex number is $\frac{1}{i-2}$. Then, the complex number is

(1) $\frac{1}{i+2}$ (2) $\frac{-1}{i+2}$ (3) $\frac{-1}{i-2}$ (4) $\frac{1}{i-2}$

5. If $z = \frac{\left(\sqrt{3}+i\right)^3 (3i+4)^2}{(8+6i)^2}$, then $|z|$ is equal to

(1) 0 (2) 1 (3) 2 (4) 3

6. If z is a non zero complex number, such that $2iz^2 = \bar{z}$ then $|z|$ is

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3

7. If $|z-2+i| \le 2$, then the greatest value of $|z|$ is

(1) $\sqrt{3}-2$ (2) $\sqrt{3}+2$ (3) $\sqrt{5}-2$ (4) $\sqrt{5}+2$

8. If $\left|z-\frac{3}{z}\right| = 2$, then the least value of $|z|$ is

(1) 1 (2) 2 (3) 3 (4) 5

9. If $|z|=1$, then the value of $\frac{1+z}{1+\bar{z}}$ is

(1) z (2) $\bar{z}$ (3) $\frac{1}{z}$ (4) 1

10. The solution of the equation $|z|-z = 1+2i$ is

(1) $\frac{3}{2}-2i$ (2) $-\frac{3}{2}+2i$ (3) $2-\frac{3}{2}i$ (4) $2+\frac{3}{2}i$

11. If $|z_1|=1,\ |z_2|=2,\ |z_3|=3$ and $|9z_1z_2+4z_1z_3+z_2z_3|=12$, then the value of $|z_1+z_2+z_3|$ is

(1) 1 (2) 2 (3) 3 (4) 4

12. If z is a complex number such that $z \in \mathbb{C}\setminus\mathbb{R}$ and $z+\frac{1}{z} \in \mathbb{R}$, then $|z|$ is

(1) 0 (2) 1 (3) 2 (4) 3

13. z_1, z_3, and z_3 are complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$ then $z_1^2 + z_2^2 + z_3^2$ is

(1) 3 (2) 2 (3) 1 (4) 0

14. If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3

15. If $z = x + iy$ is a complex number such that $|z+2| = |z-2|$, then the locus of z is

(1) real axis (2) imaginary axis (3) ellipse (4) circle

16. The principal argument of $\frac{3}{-1+i}$ is

(1) $\frac{-5\pi}{6}$ (2) $\frac{-2\pi}{3}$ (3) $\frac{-3\pi}{4}$ (4) $\frac{-\pi}{2}$

17. The principal argument of $(\sin 40° + i\cos 40°)^5$ is

(1) $-110°$ (2) $-70°$ (3) $70°$ (4) $110°$

18. If $(1+i)(1+2i)(1+3i)\cdots(1+ni) = x + iy$, then $2 \cdot 5 \cdot 10 \cdots (1+n^2)$ is

(1) 1 (2) i (3) $x^2 + y^2$ (4) $1 + n^2$

19. If $\omega \neq 1$ is a cubic root of unity and $(1+\omega)^7 = A + B\omega$, then (A, B) equals

(1) $(1,0)$ (2) $(-1,1)$ (3) $(0,1)$ (4) $(1,1)$

20. The principal argument of the complex number $\frac{\left(1+i\sqrt{3}\right)^2}{4i\left(1-i\sqrt{3}\right)}$ is

(1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{2}$

21. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is

(1) -2 (2) -1 (3) 1 (4) 2

22. The product of all four values of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is

(1) -2 (2) -1 (3) 1 (4) 2

23. If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to

(1) 1 (2) -1 (3) $\sqrt{3}i$ (4) $-\sqrt{3}i$

24. The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is

(1) $cis\frac{2\pi}{3}$ (2) $cis\frac{4\pi}{3}$ (3) $-cis\frac{2\pi}{3}$ (4) $-cis\frac{4\pi}{3}$

25. If $\omega = cis\dfrac{2\pi}{3}$, then the number of distinct roots of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

(1) 1 (2) 2 (3) 3 (4) 4

SUMMARY

In this chapter we studied

Rectangular form of a complex number is $x+iy$ (or $x+yi$), where x and y are real numbers.

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$. That is $x_1 = x_2$ and $y_1 = y_2$.

The conjugate of the complex number $x+iy$ is defined as the complex number $x-iy$.

Properties of complex conjugates

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(3) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(4) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

(5) $\text{Re}(z) = \dfrac{z+\overline{z}}{2}$

(6) $\text{Im}(z) = \dfrac{z-\overline{z}}{2i}$

(7) $\overline{(z^n)} = (\overline{z})^n$, where n is an integer

(8) z is real if and only if $z = \overline{z}$

(9) z is purely imaginary if and only if $z = -\overline{z}$

(10) $\overline{\overline{z}} = z$

If $z = x+iy$, then $\sqrt{x^2+y^2}$ is called modulus of z. It is denoted by $|z|$.

Properties of Modulus of a complex number

(1) $|z| = |\overline{z}|$

(2) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1||z_2|$

(4) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(5) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(6) $|z^n| = |z|^n$, where n is an integer

(7) $\text{Re}(z) \leq |z|$

(8) $\text{Im}(z) \leq |z|$

Formula for finding square root of a complex number

$$\sqrt{a+ib}=\pm\left(\sqrt{\frac{|z|+a}{2}}+i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right),\text{ where } z=a+ib \text{ and } b\neq 0.$$

Let r and θ be polar coordinates of the point $P(x,y)$ that corresponds to a non-zero complex number $z=x+iy$. The polar form or trigonometric form of a complex number P is

$$z=r(\cos\theta+i\sin\theta).$$

Properties of polar form

Property 1: If $z=r(\cos\theta+i\sin\theta)$, then $z^{-1}=\frac{1}{r}(\cos\theta-i\sin\theta)$.

Property 2: If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then $z_1z_2=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$.

Property3: If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then $\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$.

de Moivre's Theorem

(a) Given any complex number $\cos\theta+i\sin\theta$ and any integer n,

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

(b) If x is rational, then $\cos x\theta+i\sin x\theta$ in one of the values of $(\cos\theta+i\sin\theta)^x$

The n^{th} roots of complex number $z=r(\cos\theta+i\sin\theta)$ are

$$z^{1/n}=r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right),\ k=0,1,2,3,\ldots,n-1.$$

ICT CORNER

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Open the Browser, type the URL Link given below (or) Scan the QR code. GeoGebra work book named "12th Standard Mathematics" will open. In the left side of the work book there are many chapters related to your text book. Click on the chapter named "Complex Numbers". You can see several work sheets related to the chapter. Select the work sheet "Geometrical Meaning"

CPL9J

Chapter

3

Theory of Equations

"It seems that if one is working from the point of view of getting beauty in one's equation, and if one has really a sound insight, one is on a sure line of progress."

- Paul Dirac

3.1 Introduction

One of the oldest problems in mathematics is solving algebraic equations, in particular, finding the roots of polynomial equations. Starting from Sumerian and Babylonians around 2000 BC (BCE), mathematicians and philosophers of Egypt, Greece, India, Arabia, China, and almost all parts of the world attempted to solve polynomial equations.

Abel
(1802-1829)

The ancient mathematicians stated the problems and their solutions entirely in terms of words. They attempted particular problems and there was no generality. Brahmagupta was the first to solve quadratic equations involving negative numbers. Euclid, Diophantus, Brahmagupta, Omar Khayyam, Fibonacci, Descartes, and Ruffini were a few among the mathematicians who worked on polynomial equations. Ruffini claimed that there was no algebraic formula to find the solutions to fifth degree equations by giving a lengthy argument which was difficult to follow; finally in 1823, Norwegian mathematician Abel proved it.

Suppose that a manufacturing company wants to pack its product into rectangular boxes. It plans to construct the boxes so that the length of the base is six units more than the breadth, and the height of the box is to be the average of the length and the breadth of the base. The company wants to know all possible measurements of the sides of the box when the volume is fixed.

If we let the breadth of the base as x, then the length is $x+6$ and its height is $x+3$. Hence the volume of the box is $x(x+3)(x+6)$. Suppose the volume is 2618 cubic units, then we must have $x^3+9x^2+18x=2618$. If we are able to find an x satisfying the above equation, then we can construct a box of the required dimension.

We know a circle and a straight line cannot intersect at more than two points. But how can we prove this? Mathematical equations help us to prove such statements. The circle with centre at origin and radius r is represented by the equation $x^2+y^2=r^2$, in the xy-plane. We further know that a line, in the same plane, is given by the equation $ax+by+c=0$. The points of intersection of the circle and the straight line are the points which satisfy both equations. In other words, the solutions of the simultaneous equations

$$x^2+y^2=r^2 \text{ and } ax+by+c=0$$

give the points of intersection. Solving the above system of equations, we can conclude whether they touch each other, intersect at two points or do not intersect each other.

There are some ancient problems on constructing geometrical objects using only a compass and a ruler (straight edge without units marking). For instance, a regular hexagon and a regular polygon of 17 sides are constructible whereas a regular heptagon and a regular polygon of 18 sides are not constructible. Using only a compass and a ruler certain geometrical constructions, particularly the following three, are not possible to construct:

- Trisecting an angle (dividing a given angle into three equal angles).
- Squaring a circle (constructing a square with area of a given circle). [Srinivasa Ramanujan has given an approximate solution in his “Note Book”]
- Doubling a cube (constructing a cube with twice the volume of a given cube).

These ancient problems are settled only after converting these geometrical problems into problems on polynomials; in fact **these constructions are impossible**. Mathematics is a very nice tool to prove impossibilities.

When solving a real life problem, mathematicians convert the problem into a mathematical problem, solve the mathematical problem using known mathematical techniques, and then convert the mathematical solution into a solution of the real life problem. Most of the real life problems, when converting into a mathematical problem, end up with a mathematical equation. While discussing the problems of deciding the dimension of a box, proving certain geometrical results and proving some constructions impossible, we end up with polynomial equations.

In this chapter we learn some theory about equations, particularly about polynomial equations, and their solutions; we study some properties of polynomial equations, formation of polynomial equations with given roots, the fundamental theorem of algebra, and to know about the number of positive and negative roots of a polynomial equation. Using these ideas we reach our goal of solving polynomial equations of certain types. We also learn to solve some non–polynomial equations using techniques developed for polynomial equations.

Learning Objectives

Upon completion of this chapter, the students will be able to

- form polynomial equations satisfying given conditions on roots.
- demonstrate the techniques to solve polynomial equations of higher degree.
- solve equations of higher degree when some roots are known to be complex or surd, irrational, and rational.
- identify and solve reciprocal equations.
- determine the number of positive and negative roots of a polynomial equation using Descartes Rule.

3.2 Basics of Polynomial Equations

3.2.1 Different types of Polynomial Equations

We already know that, for any non–negative integer n, a **polynomial** of **degree** n in one variable x is an expression given by

$$P \equiv P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{... (1)}$$

where $a_r \in \mathbb{C}$ are constants, $r = 0, 1, 2, \ldots, n$ with $a_n \neq 0$. The variable x is real or complex.

When all the coefficients of a polynomial P are real, we say **"*P* is a polynomial over $\mathbb{R}$"**. Similarly we use terminologies like **"*P* is a polynomial over $\mathbb{C}$"**, **"*P* is a polynomial over $\mathbb{Q}$"**, and ***P* is a polynomial over $\mathbb{Z}$"**.

The function P defined by $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is called a **polynomial function.** The equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad \text{... (2)}$$

is called a **polynomial equation**.

If $a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 = 0$ for some $c \in \mathbb{C}$, then c is called a **zero** of the polynomial (1) and **root** or **solution** of the polynomial equation (2).

If c is a root of an equation in one variable x, we write it as "$x = c$ is a root". The constants a_r are called **coefficients.** The coefficient a_n is called the **leading coefficient** and the term $a_n x^n$ is called the **leading term**. The coefficients may be any number, real or complex. The only restriction we made is that the leading coefficient a_n is nonzero. A polynomial with the leading coefficient 1 is called a **monic polynomial**.

Remark:

We note the following:

- Polynomial functions are defined for all values of x.
- Every nonzero constant is a polynomial of degree 0.
- The constant 0 is also a polynomial called the **zero polynomial**; its degree is not defined.
- The degree of a polynomial is a nonnegative integer.
- The zero polynomial is the only polynomial with leading coefficient 0.
- Polynomials of degree two are called **quadratic polynomials**.
- Polynomials of degree three are called **cubic polynomials**.
- Polynomial of degree four are called **quartic polynomials**.

It is customary to write polynomials in descending powers of x. That is, we write polynomials having the term of highest power (leading term) as the first term and the constant term as the last term.

For instance, $2x + 3y + 4z = 5$ and $6x^2 + 7x^2 y^3 + 8z = 9$ are equations in three variables x, y, z; $x^2 - 4x + 5 = 0$ is an equation in one variable x. In the earlier classes we have solved trigonometric equations, system of linear equations, and some polynomial equations.

We know that 3 is a zero of the polynomial x^2-5x+6 and 3 is a root or solution of the equation $x^2-5x+6=0$. We note that $\cos x=\sin x$ and $\cos x+\sin x=1$ are also equations in one variable x. However, $\cos x-\sin x$ and $\cos x+\sin x-1$ are not polynomials and hence $\cos x=\sin x$ and $\cos x+\sin x=1$ are not "polynomial equations". We are going to consider only "polynomial equations" and equations which can be solved using polynomial equations in one variable.

We recall that $\sin^2 x+\cos^2 x=1$ is an identity on $\mathbb{R}$, while $\sin x+\cos x=1$ and $\sin^3 x+\cos^3 x=1$ are equations.

It is important to note that the coefficients of a polynomial can be real or complex numbers, but the exponents must be nonnegative integers. For instance, the expressions $3x^{-2}+1$ and $5x^{\frac{1}{2}}+1$ are not polynomials. We already learnt about polynomials and polynomial equations, particularly about quadratic equations. In this section let us quickly recall them and see some more concepts.

3.2.2 Quadratic Equations

For the quadratic equation $ax^2+bx+c=0$, b^2-4ac is called the discriminant and it is usually denoted by Δ. We know that $\frac{-b+\sqrt{\Delta}}{2a}$ and $\frac{-b-\sqrt{\Delta}}{2a}$ are roots of the $ax^2+bx+c=0$. The two roots together are usually written as $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. It is unnecessary to emphasize that $a\neq 0$, since by saying that ax^2+bx+c is a quadratic polynomial, it is implied that $a\neq 0$.

We also learnt that $\Delta=0$ if, and only if, the roots are equal. When a,b,c are real, we know

- $\Delta>0$ if, and only if, the roots are real and distinct
- $\Delta<0$ if, and only if, the quadratic equation has no real roots.

3.3 Vieta's Formulae and Formation of Polynomial Equations

Vieta's formulae relate the coefficients of a polynomial to sums and products of its roots. Vieta was a French mathematician whose work on polynomials paved the way for modern algebra.

3.3.1 Vieta's formula for Quadratic Equations

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Let α and β be the roots of the quadratic equation $ax^2+bx+c=0$. Then

$$ax^2+bx+c=a(x-\alpha)(x-\beta)=ax^2-a(\alpha+\beta)x+a(\alpha\beta)=0.$$

Equating the coefficients of like powers, we see that

$$\alpha+\beta=\frac{-b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

So **a** quadratic equation whose roots are α and β is $x^2-(\alpha+\beta)x+\alpha\beta=0$; that is, **a** quadratic equation with given roots is,

$$x^2-(\text{sum of the roots})\,x+\text{ product of the roots}=0. \qquad \ldots (1)$$

Note

The indefinite article **a** is used in the above statement. In fact, if $P(x)=0$ is a quadratic equation whose roots are α and β, then $cP(x)$ is also a quadratic equation with roots α and β for any non-zero constant c.

In earlier classes, using the above relations between roots and coefficients we constructed a quadratic equation, having α and β as roots. In fact, such an equation is given by (1). For instance, a quadratic equation whose roots are 3 and 4 is given by $x^2-7x+12=0$.

Further we construct new polynomial equations whose roots are functions of the roots of a given polynomial equation; in this process we form a new polynomial equation without finding the roots of the given polynomial equation. For instance, we construct a polynomial equation by increasing the roots of a given polynomial equation by two as given below.

Example 3.1

If α and β are the roots of the quadratic equation $17x^2+43x-73=0$, construct a quadratic equation whose roots are $\alpha+2$ and $\beta+2$.

Solution

Since α and β are the roots of $17x^2+43x-73=0$, we have $\alpha+\beta=\frac{-43}{17}$ and $\alpha\beta=\frac{-73}{17}$.

We wish to construct a quadratic equation with roots $\alpha+2$ and $\beta+2$. Thus, to construct such a quadratic equation, calculate,

$$\text{the sum of the roots} = \alpha+\beta+4 = \frac{-43}{17}+4 = \frac{25}{17} \quad \text{and}$$

$$\text{the product of the roots} = \alpha\beta+2(\alpha+\beta)+4 = \frac{-73}{17}+2\left(\frac{-43}{17}\right)+4 = \frac{-91}{17}.$$

Hence a quadratic equation with required roots is $x^2-\frac{25}{17}x-\frac{91}{17} = 0$.

Multiplying this equation by 17, gives $17x^2-25x-91 = 0$

which is also a quadratic equation having roots $\alpha+2$ and $\beta+2$. ■

Example 3.2

If α and β are the roots of the quadratic equation $2x^2-7x+13=0$, construct a quadratic equation whose roots are α^2 and β^2.

Solution

Since α and β are the roots of the quadratic equation, we have $\alpha+\beta = \frac{7}{2}$ and $\alpha\beta=\frac{13}{2}$.

Thus, to construct a new quadratic equation,

$$\text{Sum of the roots} = \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{-3}{4}.$$

$$\text{Product of the roots} = \alpha^2\beta^2=(\alpha\beta)^2=\frac{169}{4}$$

Thus a required quadratic equation is $x^2+\frac{3}{4}x+\frac{169}{4}=0$. From this we see that

$$4x^2+3x+169 = 0$$

is a quadratic equation with roots α^2 and β^2. ■

Remark

In Examples 3.1 and 3.2, we have computed the sum and the product of the roots using the known $\alpha+\beta$ and $\alpha\beta$. In this way we can construct quadratic equation with desired roots, provided the sum and the product of the roots of a new quadratic equation can be written using the sum and the product of the roots of the given quadratic equation. We note that we have not solved the given equation; we do not know the values of α and β even after completing the task.

3.3.2 Vieta's formula for Polynomial Equations

What we have learnt for quadratic polynomial, can be extended to polynomials of higher degree. In this section we study the relations of the zeros of a polynomial of higher degree with its coefficients. We also learn how to form polynomials of higher degree when some information about the zeros are known. In this chapter, we use either zeros of a polynomial of degree n or roots of polynomial equation of degree n .

3.3.2 (a) The Fundamental Theorem of Algebra

If a is a root of a polynomial equation $P(x)=0$, then $(x-a)$ is a **factor** of $P(x)$. So, deg $(P(x)) \geq 1$. If a and b are roots of $P(x)=0$ then $(x-a)(x-b)$ is a factor of $P(x)$ and hence deg $(P(x)) \geq 2$. Similarly if $P(x)=0$ has n roots, then its degree must be greater than or equal to n. In other words, a **polynomial equation of degree n cannot have more than n roots**.

In earlier classes we have learnt about "multiplicity". Let us recall what we mean by "multiplicity". We know if $(x-a)^k$ is a factor of a polynomial equation $P(x)=0$ and $(x-a)^{k+1}$ is not a factor of the polynomial equation, $P(x)=0$, then a is called a root of **multiplicity** k . For instance, 3 is a root of multiplicity 2 for the equation $x^2-6x+9=0$ and $x^3-7x^2+159x-9=0$. Though we are not going to use complex numbers as coefficients, it is worthwhile to mention that the imaginary number $2+i$ is a root of multiplicity 2 for the polynomials $x^2-(4+2i)x+3+4i=0$ and $x^4-8x^3+26x^2-40x+25=0$. If a is a root of multiplicity 1 for a polynomial equation, then a is called a **simple root** of the polynomial equation.

If $P(x)=0$ has n roots counted with multiplicity, then also, we see that its degree must be greater than or equal to n . In other words, "a polynomial equation of degree n cannot have more than n roots, even if the roots are counted with their multiplicities".

One of the important theorems in the theory of equations is the fundamental theorem of algebra. As the proof is beyond the scope of the Course, we state it without proof.

Theorem 3.1 (The Fundamental Theorem of Algebra)

Every polynomial equation of degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Using this, we can prove that a polynomial equation of degree n has at least n roots in $\mathbb{C}$ when the roots are counted with their multiplicities. This statement together with our discussion above says that

a polynomial equation of degree n has exactly n roots in $\mathbb{C}$ when the roots are counted with their multiplicities.

Some authors state this statement as the fundamental theorem of algebra.

3.3.2(b) Vieta's Formula

(i) Vieta's Formula for Polynomial equation of degree 3

Now we obtain these types of relations to higher degree polynomials. Let us consider a general cubic equation

$$ax^3+bx^2+cx+d = 0 .$$

By the fundamental theorem of algebra, it has three roots. Let α , β , and γ be the roots. Thus we have

$$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$$

Expanding the right hand side,

$$ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\gamma\alpha)x - a(\alpha\beta\gamma).$$

Comparing the coefficients of like powers, we obtain

$$\alpha+\beta+\gamma = \frac{-b}{a}, \quad \alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a} \text{ and } \alpha\beta\gamma = \frac{-d}{a}.$$

Since the degree of the polynomial equation is 3, we have $a \neq 0$ and hence division by a is meaningful. If a monic cubic polynomial has roots α, β, and γ, then

$$\begin{aligned} \text{coefficient of } x^2 &= -(\alpha+\beta+\gamma), \\ \text{coefficient of } x &= \alpha\beta+\beta\gamma+\gamma\alpha, \text{ and} \\ \text{constant term} &= -\alpha\beta\gamma. \end{aligned}$$

(ii) Vieta's Formula for Polynomial equation of degree $n > 3$

The same is true for higher degree monic polynomial equations as well. If a monic polynomial equation of degree n has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, then

$$\boxed{\begin{aligned} &\text{coefficient of } x^{n-1} && = \textstyle\sum_1 && = -\textstyle\sum \alpha_1 \\ &\text{coefficient of } x^{n-2} && = \textstyle\sum_2 && = \textstyle\sum \alpha_1\alpha_2 \\ &\text{coefficient of } x^{n-3} && = \textstyle\sum_3 && = -\textstyle\sum \alpha_1\alpha_2\alpha_3 \\ &\text{coefficient of } x && = \textstyle\sum_{n-1} && = (-1)^{n-1}\textstyle\sum \alpha_1\alpha_2\ldots\alpha_{n-1} \\ &\text{coefficient of } x^0 = \text{constant term} && = \textstyle\sum_n && = (-1)^n \alpha_1\alpha_2\ldots\alpha_n \end{aligned}}$$

where $\sum \alpha_1$ denotes the sum of all roots, $\sum \alpha_1\alpha_2$ denotes the sum of product of all roots taken two at a time, $\sum \alpha_1\alpha_2\alpha_3$ denotes the sum of product of all roots taken three at a time, and so on. If α, β, γ, and δ are the roots of a quartic equation, then $\sum \alpha_1$ is written as $\sum \alpha$, $\sum \alpha_1\alpha_2$ is written as $\sum \alpha\beta$ and so on. Thus we have,

$$\begin{aligned} \textstyle\sum \alpha &= \alpha+\beta+\gamma+\delta \\ \textstyle\sum \alpha\beta &= \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta \\ \textstyle\sum \alpha\beta\gamma &= \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta \\ \textstyle\sum \alpha\beta\gamma\delta &= \alpha\beta\gamma\delta \end{aligned}$$

When the roots are available in explicit numeric form, then also we use these convenient notations. We have to be careful when handling roots of higher multiplicity. For instance, if the roots of a cubic equation are 1, 2, 2, then $\sum \alpha = 5$ and $\sum \alpha\beta = (1\times 2)+(1\times 2)+(2\times 2) = 8$.

From the above discussion, we note that for a monic polynomial equation, the sum of the roots is the coefficient of x^{n-1} multiplied by (-1) and the product of the roots is the constant term multiplied by $(-1)^n$.

Example 3.3

If α, β, and γ are the roots of the equation $x^3+px^2+qx+r=0$, find the value of $\sum \frac{1}{\beta\gamma}$ in terms of the coefficients.

Solution

Since α, β, and γ are the roots of the equation $x^3+px^2+qx+r=0$, we have

$$\sum\nolimits_1 \quad \alpha+\beta+\gamma = -p \text{ and } \sum\nolimits_3 \quad \alpha\beta\gamma = -r.$$

Now

$$\sum \frac{1}{\beta\gamma} = \frac{1}{\beta\gamma}+\frac{1}{\gamma\alpha}+\frac{1}{\alpha\beta} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}.$$

3.3.2 (c) Formation of Polynomial Equations with given Roots

We have constructed quadratic equations when the roots are known. Now we learn how to form polynomial equations of higher degree when roots are known. How do we find a polynomial equation of degree n with roots $\alpha_1, \alpha_2, \cdots, \alpha_n$? One way of writing a polynomial equation is multiplication of the factors. That is

$$(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n)=0$$

is a polynomial equation with roots $\alpha_1, \alpha_2, \cdots, \alpha_n$. But it is not the usual way of writing a polynomial equation. We have to write the polynomial equation in the standard form which involves more computations. But by using the relations between roots and coefficients, we can write the polynomial equation directly; moreover, it is possible to write the coefficient of any particular power of x without finding the entire polynomial equation.

A cubic polynomial equation whose roots are α, β, and γ is

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0.$$

A polynomial equation of degree n with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ is given by

$$x^n-\left(\sum \alpha_1\right)x^{n-1}+\left(\sum \alpha_1\alpha_2\right)x^{n-2}-\left(\sum \alpha_1\alpha_2\alpha_3\right)x^{n-3}+\cdots+(-1)^n\alpha_1\alpha_2\cdots\alpha_n=0$$

where, $\sum \alpha_1, \sum \alpha_1\alpha_2, \sum \alpha_1\alpha_2\alpha_3, \ldots$ are as defined earlier.

For instance, a polynomial equation with roots 1, -2, and 3 is given by

$$x^3-(1-2+3)x^2+(1\times(-2)+(-2)\times 3+3\times 1)x-1\times(-2)\times 3=0$$

which, on simplification, becomes $x^3-2x^2-5x+6=0$. It is interesting to verify that the expansion of $(x-1)(x+2)(x-3)=0$ is $x^3-2x^2-5x+6=0$.

Example 3.4

Find the sum of the squares of the roots of $ax^4+bx^3+cx^2+dx+e = 0$, $a \neq 0$

Solution

Let $\alpha, \beta, \gamma,$ and δ be the roots of $ax^4+bx^3+cx^2+dx+e = 0$.

Then, we get

$$\Sigma_1 = \alpha+\beta+\gamma+\delta = -\frac{b}{a},$$

$$\Sigma_2 = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a},$$

$$\Sigma_3 = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a},$$

$$\Sigma_4 = \alpha\beta\gamma\delta = \frac{e}{a}.$$

We have to find $\alpha^2+\beta^2+\gamma^2+\delta^2$.

Applying the algebraic identity

$$(a+b+c+d)^2 \equiv a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd),$$

we get

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

$$= \left(-\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)$$

$$= \frac{b^2-2ac}{a^2}.$$

Example 3.5

Find the condition that the roots of cubic equation $x^3+ax^2+bx+c = 0$ are in the ratio $p:q:r$.

Solution

Since roots are in the ratio $p:q:r$, we can assume the roots as $p\lambda, q\lambda$ and $r\lambda$.

Then, we get

$$\Sigma_1 = p\lambda+q\lambda+r\lambda = -a, \quad \text{....(1)}$$

$$\Sigma_2 = (p\lambda)(q\lambda)+(q\lambda)(r\lambda)+(r\lambda)(p\lambda) = b, \quad \text{....(2)}$$

$$\Sigma_3 = (p\lambda)(q\lambda)(r\lambda) = -c, \quad \text{....(3)}$$

Now, we get

$$(1) \Rightarrow \lambda = -\frac{a}{p+q+r} \quad \text{....(4)}$$

$$(3) \Rightarrow \lambda^3 = -\frac{c}{pqr} \quad \text{.....(5)}$$

Substituting (4) in (5), we get

$$\left(-\frac{a}{p+q+r}\right)^3 = -\frac{c}{pqr} \Rightarrow pqra^3 = c(p+q+r)^3.$$

Example 3.6

Form the equation whose roots are the squares of the roots of the cubic equation $x^3+ax^2+bx+c = 0$.

Solution

Let α, β, and γ be the roots of $x^3+ax^2+bx+c = 0$.

Then, we get

$$\sum\nolimits_1 = \alpha+\beta+\gamma = -a, \qquad \text{....(1)}$$

$$\sum\nolimits_2 = \alpha\beta+\beta\gamma+\gamma\alpha = b, \qquad \text{....(2)}$$

$$\sum\nolimits_3 = \alpha\beta\gamma = -c. \qquad \text{....(3)}$$

We have to form the equation whose roots are α^2, β^2, and γ^2.
Using (1), (2) and (3), we find the following:

$$\sum\nolimits_1 = \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha) = (-a)^2-2(b) = a^2-2b,$$

$$\sum\nolimits_2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2((\alpha\beta)(\beta\gamma)+(\beta\gamma)(\gamma\alpha)+(\gamma\alpha)(\alpha\beta))$$

$$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\beta+\gamma+\alpha) = (b)^2-2(-c)(-a) = b^2-2ca$$

$$\sum\nolimits_3 = \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = (-c)^2 = c^2.$$

Hence, the required equation is

$$x^3-\left(\alpha^2+\beta^2+\gamma^2\right)x^2+\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)x-\alpha^2\beta^2\gamma^2 = 0.$$

That is, $x^3-\left(a^2-2b\right)x^2+\left(b^2-2ca\right)x-c^2=0.$ ■

Example 3.7

If p is real, discuss the nature of the roots of the equation $4x^2+4px+p+2=0$, in terms of p.

Solution

The discriminant $\Delta=(4p)^2-4(4)(p+2)=16\left(p^2-p-2\right)=16(p+1)(p-2)$. So, we get

$$\Delta<0 \text{ if } -1<p<2$$

$$\Delta=0 \text{ if } p=-1 \text{ or } p=2$$

$$\Delta>0 \text{ if } -\infty<p<-1 \text{ or } 2<p<\infty$$

Thus the given polynomial has

imaginary roots if $-1<p<2$;

equal real roots if $p=-1$ or $p=2$;

distinct real roots if $-\infty<p<-1$ or $2<p<\infty$. ■

EXERCISE 3.1

1. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.
2. Construct a cubic equation with roots
   (i) $1, 2$, and 3 (ii) $1, 1$, and -2 (iii) $2, \frac{1}{2}$ and 1.
3. If α, β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equation whose roots are
   (i) $2\alpha, 2\beta, 2\gamma$ (ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ (iii) $-\alpha, -\beta, -\gamma$
4. Solve the equation $3x^3-16x^2+23x-6=0$ if the product of two roots is 1.
5. Find the sum of squares of roots of the equation $2x^4-8x^3+6x^2-3=0$.

6. Solve the equation $x^3-9x^2+14x+24=0$ if it is given that two of its roots are in the ratio $3:2$.

7. If α,β, and γ are the roots of the polynomial equation $ax^3+bx^2+cx+d=0$, find the value of $\sum\frac{\alpha}{\beta\gamma}$ in terms of the coefficients.

8. If α,β,γ, and δ are the roots of the polynomial equation $2x^4+5x^3-7x^2+8=0$, find a quadratic equation with integer coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.

9. If p and q are the roots of the equation $lx^2+nx+n=0$, show that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$.

10. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be equal to $\frac{pq'-p'q}{q-q'}$ or $\frac{q-q'}{p'-p}$.

11. A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was left standing.

3.4 Nature of Roots and Nature of Coefficients of Polynomial Equations

3.4.1 Imaginary Roots

For a quadratic equation with real coefficients, if $\alpha+i\beta$ is a root, then $\alpha-i\beta$ is also a root. In this section we shall prove that this is true for higher degree polynomials as well.

We now prove one of the very important theorems in the theory of equations.

Theorem 3.2 (Complex Conjugate Root Theorem)

If a complex number z_0 is a root of a polynomial equation with real coefficients, then its complex conjugate $\overline{z}_0$ is also a root.

Proof

Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_o=0$ be a polynomial equation with real coefficients. Let z_0 be a root of this polynomial equation. So, $P(z_0)=0$. Now

$$\begin{aligned}
P(\overline{z_0}) &= a_n\overline{z}_0^n+a_{n-1}\overline{z}_0^{n-1}+\cdots+a_1\overline{z}_0+a_0\\
&= a_n\overline{z_0^n}+a_{n-1}\overline{z_0^{n-1}}+\cdots+a_1\overline{z_0}+a_0\\
&= \overline{a_n}\,\overline{z_0^n}+\overline{a_{n-1}}\,\overline{z_0^{n-1}}+\cdots+\overline{a_1}\,\overline{z_0}+\overline{a_0} \qquad (a_r=\overline{a_r} \text{ as } a_r \text{ is real for all } r)\\
&= \overline{a_nz_0^n}+\overline{a_{n-1}z_0^{n-1}}+\cdots+\overline{a_1z_0}+\overline{a_0}\\
&= \overline{a_nz_0^n+a_{n-1}z_0^{n-1}+\cdots+a_1z_0+a_0} = \overline{P(z_0)}=\overline{0}=0.
\end{aligned}$$

That is $P(\overline{z}_0)=0$; this implies that whenever z_0 is a root (i.e. $P(z_0)$=0), its conjugate $\overline{z}_0$ is also a root. ■

If one asks whether 2 is a complex number, many students hesitate to say "yes". As every integer is a rational number, we know that every real number is also a complex number. So to clearly specify a complex number that is not a real number, that is to specify numbers of form $\alpha + i\beta$ with $\beta \neq 0$, we use the term **"non-real complex number".** Some authors call such a number an **imaginary number**.

Remark 1

Let $z_0 = \alpha + i\beta$ with $\beta \neq 0$. Then $\bar{z}_0 = \alpha - i\beta$. If $\alpha + i\beta$ is a root of a polynomial equation $P(x) = 0$ with real coefficients, then by Complex Conjugate Root Theorem, $\alpha - i\beta$ is also a root of $P(x) = 0$. Usually the above statement will be stated as **complex roots occur in pairs;** but actually it means that **non-real complex roots or imaginary roots occur as conjugate pairs, being the coefficients of the polynomial equation are real.**

Remark 2

From this we see that any odd degree polynomial equation with real coefficients has at least one real root; in fact, the number of real roots of an odd degree polynomial equation with real coefficients is always an odd number. Similarly the number of real roots of an even degree polynomial equation with real coefficients is always an even number.

Example 3.8

Find the monic polynomial equation of minimum degree with real coefficients having $2 - \sqrt{3}\,i$ as a root.

Solution

Since $2 - \sqrt{3}\,i$ is a root of the required polynomial equation with real coefficients, $2 + \sqrt{3}\,i$ is also a root. Hence the sum of the roots is 4 and the product of the roots is 7. Thus $x^2 - 4x + 7 = 0$ is the required monic polynomial equation. ■

3.4.2 Irrational Roots

If we further restrict the coefficients of the quadratic equation $ax^2 + bx + c = 0$ to be rational, we get some interesting results. Let us consider a quadratic equation $ax^2 + bx + c = 0$ with a, b, and c rational. As usual let $\Delta = b^2 - 4ac$ and let r_1 and r_2 be the roots. In this case, when $\Delta = 0$, we have $r_1 = r_2$; this root is not only real, it is in fact a rational number.

When Δ is positive, then no doubt that $\sqrt{\Delta}$ exists in $\mathbb{R}$ and we get two distinct real roots. But $\sqrt{\Delta}$ will be a rational number for certain values of a, b, and c, and it is an irrational number for other values of a, b, and c.

If $\sqrt{\Delta}$ is rational, then both r_1 and r_2 are rational.

If $\sqrt{\Delta}$ is irrational, then both r_1 and r_2 are irrational.

Immediately we have a question. If $\Delta > 0$, when will $\sqrt{\Delta}$ be rational and when will it be irrational? To answer this question, first we observe that Δ is rational, as the coefficients are rational numbers. So $\Delta = \dfrac{m}{n}$ for some positive integers m and n with $(m, n) = 1$ where (m, n) denotes the greatest common divisor of m and n. It is now easy to understand that $\sqrt{\Delta}$ is rational if and only if both m and n are perfect squares. Also, $\sqrt{\Delta}$ is irrational if and only if at least one of m and n is not a perfect square.

We are familiar with irrational numbers of the type $p+\sqrt{q}$ where p and q are rational numbers and $\sqrt{q}$ is irrational. Such numbers are called **surds**. As in the case of imaginary roots, we can prove that if $p+\sqrt{q}$ is a root of a polynomial, then $p-\sqrt{q}$ is also a root of the same polynomial equation, when all the coefficients are rational numbers. Though this is true for polynomial equation of any degree and can be proved using the technique used in the proof of imaginary roots, we state and prove this only for a quadratic equation in Theorem 3.3.

Before proving the theorem, we recall that if a and b are rational numbers and c is an irrational number such that $a+bc$ is a rational number, then b must be 0; further if $a+bc=0$, then a and b must be 0.

For instance, if $a+b\sqrt{2}\in\mathbb{Q}$, then b must be 0, and if $a+b\sqrt{2}=0$ then $a=b=0$. Now we state and prove a general result as given below.

Theorem 3.3

Let p and q be rational numbers such that $\sqrt{q}$ is irrational. If $p+\sqrt{q}$ is a root of a quadratic equation with rational coefficients, then $p-\sqrt{q}$ is also a root of the same equation.

Proof

We prove the theorem by assuming that the quadratic equation is a monic polynomial equation. The result for non-monic polynomial equation can be proved in a similar way.

Let p and q be rational numbers such that $\sqrt{q}$ is irrational. Let $p+\sqrt{q}$ be a root of the equation $x^2+bx+c=0$ where b and c are rational numbers.

Let α be the other root. Computing the sum of the roots, we get

$$\alpha+p+\sqrt{q} = -b$$

and hence $\alpha+\sqrt{q}=-b-p\in\mathbb{Q}$. Taking $-b-p$ as s, we have $\alpha+\sqrt{q}=s$.

This implies that

$$\alpha = s-\sqrt{q}.$$

Computing the product of the roots, we get

$$(s-\sqrt{q})(p+\sqrt{q}) = c$$

and hence $(sp-q)+(s-p)\sqrt{q}=c\in\mathbb{Q}$. Thus $s-p=0$. This implies that $s=p$ and hence we get $\alpha=p-\sqrt{q}$. So, the other root is $p-\sqrt{q}$. ■

Remark

The statement of Theorem 3.3 may seem to be a little bit complicated. We should not be in a hurry to make the theorem short by writing "**for a polynomial equation with rational coefficients, irrational roots occur in pairs**". This is not true.

For instance, the equation $x^3 - 2 = 0$ has only one irrational root, namely $\sqrt[3]{2}$. Of course, the other two roots are imaginary numbers (What are they?).

Example 3.9

Find a polynomial equation of minimum degree with rational coefficients, having $2-\sqrt{3}$ as a root.

Solution

Since $2-\sqrt{3}$ is a root and the coefficients are rational numbers, $2+\sqrt{3}$ is also a root. A required polynomial equation is given by

$$x^2 - (\text{Sum of the roots})\ x + \text{Product of the roots} = 0$$

and hence

$$x^2 - 4x + 1 = 0$$

is a required equation. ■

Note

We note that the term "rational coefficients" is very important; otherwise, $x-(2-\sqrt{3})=0$ will be a polynomial equation which has $2-\sqrt{3}$ as a root but not $2+\sqrt{3}$. We state the following result without proof.

Theorem 3.4

Let p and q be rational numbers so that $\sqrt{p}$ and $\sqrt{q}$ are irrational numbers; further let one of $\sqrt{p}$ and $\sqrt{q}$ be not a rational multiple of the other. If $\sqrt{p}+\sqrt{q}$ is a root of a polynomial equation with rational coefficients, then $\sqrt{p}-\sqrt{q}, -\sqrt{p}+\sqrt{q}$, and $-\sqrt{p}-\sqrt{q}$ are also roots of the same polynomial equation.

Example 3.10

Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Solution

Since $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a root, $x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a factor. To remove the outermost square root, we take $x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as another factor and find their product

$$\left(x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right) = x^2 - \frac{\sqrt{2}}{\sqrt{3}}.$$

Still we didn't achieve our goal. So we include another factor $x^2+\frac{\sqrt{2}}{\sqrt{3}}$ and get the product

$$\left(x^2-\frac{\sqrt{2}}{\sqrt{3}}\right)\left(x^2+\frac{\sqrt{2}}{\sqrt{3}}\right) = x^4 - \frac{2}{3}.$$

So, $3x^4 - 2 = 0$ is a required polynomial equation with the integer coefficients. ■

Now we identify the nature of roots of the given equation without solving the equation. The idea comes from the negativity, equal to 0 and positivity of $\Delta = b^2 - 4ac$.

3.4.3 Rational Roots

If all the coefficients of a quadratic equation are integers, then Δ is an integer, and when it is positive, we have, $\sqrt{\Delta}$ is rational if, and only if, Δ is a perfect square. In other words, the equation $ax^2+bx+c=0$ with integer coefficients has rational roots, if, and only if, Δ is a perfect square.

What we discussed so far on polynomial equations of rational coefficients holds for polynomial equations with integer coefficients as well. In fact, multiplying the polynomial equation with rational coefficients, by a common multiple of the denominators of the coefficients, we get a polynomial equation of integer coefficients having the same roots. Of course, we have to handle this situation carefully. For instance, there is a monic polynomial equation of degree 1 with rational coefficients having $\frac{1}{2}$ as a root, whereas there is no monic polynomial equation of any degree with integer coefficients having $\frac{1}{2}$ as a root.

Example 3.11

Show that the equation $2x^2-6x+7=0$ cannot be satisfied by any real values of x.

Solution

$\Delta=b^2-4ac=-20<0$. The roots are imaginary numbers. ■

Example 3.12

If $x^2+2(k+2)x+9k=0$ has equal roots, find k.

Solution

Here $\Delta=b^2-4ac=0$ for equal roots. This implies $4(k+2)^2=4(9)k$.This implies $k=4$ or 1. ■

Example 3.13

Show that, if p,q,r are rational, the roots of the equation $x^2-2px+p^2-q^2+2qr-r^2=0$ are rational.

Solution

The roots are rational if $\Delta=b^2-4ac=(-2p)^2-4\left(p^2-q^2+2qr-r^2\right)$.

But this expression reduces to $4\left(q^2-2qr+r^2\right)$ or $4(q-r)^2$ which is a perfect square. Hence the roots are rational. ■

3.5 Applications of Polynomial Equation in Geometry

Certain geometrical properties are proved using polynomial equations. We discuss a few geometric properties here.

Example 3.14

Prove that a line cannot intersect a circle at more than two points.

Solution

By choosing the coordinate axes suitably, we take the equation of the circle as $x^2+y^2=r^2$ and the equation of the straight line as $y=mx+c$. We know that the points of intersections of the circle and the straight line are the points which satisfy the simultaneous equations

$$x^2+y^2 = r^2 \qquad \text{... (1)}$$

$$y = mx + c \qquad \text{... (2)}$$

If we substitute $mx + c$ for y in (1), we get

$$x^2 + (mx + c)^2 - r^2 = 0$$

which is same as the quadratic equation

$$(1 + m^2)x^2 + 2mcx + (c^2 - r^2) = 0 . \qquad \text{... (3)}$$

This equation cannot have more than two solutions, and hence a line and a circle cannot intersect at more than two points. ■

It is interesting to note that a substitution makes the problem of solving a system of two equations in two variables into a problem of solving a quadratic equation.

Further we note that as the coefficients of the reduced quadratic polynomial are real, either both roots are real or both imaginary. If both roots are imaginary numbers, we conclude that the circle and the straight line do not intersect. In the case of real roots, either they are distinct or multiple roots of the polynomial. If they are distinct, substituting in (2), we get two values for y and hence two points of intersection. If we have equal roots, we say the straight line touches the circle as a tangent. As the polynomial (3) cannot have only one simple real root, a line cannot cut a circle at only one point.

Note

A technique similar to the one used in example 3.14 may be adopted to prove

- two circles cannot intersect at more than two points.
- a circle and an ellipse cannot intersect at more than four points.

EXERCISE 3.2

1. If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, in terms of k.
2. Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}i$ as a root.
3. Find a polynomial equation of minimum degree with rational coefficients, having $2i + 3$ as a root.
4. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root.
5. Prove that a straight line and parabola cannot intersect at more than two points.

3.6 Roots of Higher Degree Polynomial Equations

We know that the equation $P(x) = 0$ is called a polynomial equation. The root or zero of a polynomial equation and the solution of the corresponding polynomial equation are the same. So we use both the terminologies.

We know that it is easy to verify whether a number is a root of a polynomial equation or not, just by substitution. But when finding the roots, the problem is simple if the equation is quadratic and it is in general not so easy for a polynomial equation of higher degree.

A solution of a polynomial equation written only using its coefficients, the four basic arithmetic operators (addition, multiplication, subtraction and division), and rational exponentiation (power to a rational number, such as square, cube, square roots, cube roots and so on) is called a **radical solution**. Abel proved that it is impossible to write a radical solution for general polynomial equation of degree five or more.

EPBK6

We state a few results about polynomial equations that are useful in solving higher degree polynomial equations.

- Every polynomial in one variable is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.
- For a polynomial equation $P(x)=0$ of even degree, $P(x)\to\infty$ as $P(x)\to\pm\infty$. Thus the graph of an even degree polynomial start from left top and ends at right top.
- All results discussed on "graphing functions" in Volume I of eleventh standard textbook can be applied to the graphs of polynomials. For instance, a change in the constant term of a polynomial moves its graph up or down only.
- Every polynomial is differentiable any number of times.
- The real roots of a polynomial equation $P(x)=0$ are the points on the x-axis where the graph of $P(x)=0$ cuts the x-axis.
- If a and b are two real numbers such that $P(a)$ and $P(b)$ are of opposite signs, then
    - there is a point c on the real line for which $P(c)=0$.
    - that is, there is a root between a and b.
    - it is not necessary that there is only one root between such points; there may be $3,5,7,\ldots$ roots; that is the number of real roots between a and b is odd and not even.

However, if some information about the roots are known, then we can try to find the other roots. For instance, if it is known that two of the roots of a polynomial equation of degree 6 with rational coefficients are $2+3i$ and $4-\sqrt{5}$, then we can immediately conclude that $2-3i$ and $4+\sqrt{5}$ are also roots of the polynomial equation. So dividing by the corresponding factors, we can reduce the problems into a problem of solving a second degree equation. In this section we learn some ways of finding roots of higher degree polynomials when we have some information.

3.7 Polynomials with Additional Information

Now we discuss a few additional information with which we can solve higher degree polynomials. Sometimes the additional information will directly be given, like, one root is $2+3i$. Sometimes the additional information like, sum of the coefficients is zero, have to be found by observation of the polynomial.

3.7.1 Imaginary or Surds Roots

If $\alpha+i\beta$ is an imaginary root of a quartic polynomial with real coefficients, then $\alpha-i\beta$ is also a root; thus $(x-(\alpha+i\beta))$ and $(x-(\alpha-i\beta))$ are factors of the polynomial; hence their product is a factor; in other words, $x^2-2\alpha x+\alpha^2+\beta^2$ is a factor; we can divide the polynomial with this factor and get the second degree quotient which can be solved by known techniques; using this we can find all the roots of the polynomial.

If $2+\sqrt{3}$ is a root of a quadric polynomial equation with rational coefficients, then $2-\sqrt{3}$ is also a root; thus their product $(x-(2+\sqrt{3}))(x-(2-\sqrt{3}))$ is a factor; that is x^2-4x+1 is a factor; we can divide the polynomial with this factor and get the quotient as a second degree factor which can be solved by known techniques. Using this, we can find all the roots of the quadric equation. This technique is applicable for all surds taken in place of $2+\sqrt{3}$.

If an imaginary root and a surd root of a sixth degree polynomial with rational coefficient are known, then step by step we may reduce the problem of solving the sixth degree polynomial equation into a problem of solving a quadratic equation.

Example 3.15

If $2+i$ and $3-\sqrt{2}$ are roots of the equation

$$x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0,$$

find all roots.

Solution

Since the coefficient of the equations are all rational numbers, and $2+i$ and $3-\sqrt{2}$ are roots, we get $2-i$ and $3+\sqrt{2}$ are also roots of the given equation. Thus $(x-(2+i))$, $(x-(2-i))$, $(x-(3-\sqrt{2}))$ and $(x-(3+\sqrt{2}))$ are factors. Thus their product

$$((x-(2+i))(x-(2-i))(x-(3-\sqrt{2}))(x-(3+\sqrt{2}))$$

is a factor of the given polynomial equation. That is,

$$(x^2-4x+5)(x^2-6x+7)$$

is a factor. Dividing the given polynomial equation by this factor, we get the other factor as (x^2-3x-4) which implies that 4 and -1 are the other two roots. Thus

$$2+i, 2-i, 3+\sqrt{2}, 3-\sqrt{2}, -1, \text{ and } 4$$

are the roots of the given polynomial equation. ■

3.7.2 Polynomial equations with Even Powers Only

If $P(x)$ is a polynomial equation of degree $2n$, having only even powers of x, (that is, coefficients of odd powers are 0) then by replacing x^2 by y, we get a polynomial equation with degree n in y; let $y_1, y_2, \cdots y_n$ be the roots of this polynomial equation. Then considering the n equations $x^2 = y_r$, we can find two values for x for each y_r; these $2n$ numbers are the roots of the given polynomial equation in x.

Example 3.16

Solve the equation $x^4-9x^2+20=0$.

Solution

The given equation is

$$x^4-9x^2+20=0.$$

This is a fourth degree equation. If we replace x^2 by y, then we get the quadratic equation

$$y^2-9y+20=0.$$

It is easy to see that 4 and 5 as solutions for $y^2-9y+20=0$. Now taking $x^2=4$ and $x^2=5$, we get $2,-2,\sqrt{5},-\sqrt{5}$ as solutions of the given equation. ■

We note that the technique adopted above can be applied to polynomial equations like $x^6-17x^3+30=0$, $ax^{2k}+bx^k+c=0$ and in general polynomial equations of the form $a_nx^{kn}+a_{n-1}x^{k(n-1)}+\cdots+a_1x^k+a_0=0$ where k is any positive integer.

3.7.3 Zero Sum of all Coefficients

Let $P(x)=0$ be a polynomial equation such that the sum of the coefficients is zero. What actually the sum of coefficients is? The sum of coefficients is nothing but $P(1)$. The sum of all coefficients is zero means that $P(1)=0$ which says that 1 is a root of $P(x)$. The rest of the problem of solving the equation is easy.

Example 3.17

Solve the equation $x^3-3x^2-33x+35=0$.

Solution

The sum of the coefficients of the polynomial is 0. Hence 1 is a root of the polynomial. To find other roots, we divide $x^3-3x^2-33x+35$ by $x-1$ and get $x^2-2x-35$ as the quotient. Solving this we get 7 and -5 as roots. Thus $1,7,-5$ form the solution set of the given equation. ■

3.7.4 Equal Sums of Coefficients of Odd and Even Powers

Let $P(x)=0$ be a polynomial equation such that the sum of the coefficients of the odd powers and that of the even powers are equal. What does actually this mean? If a is the coefficient of an odd degree in $P(x)=0$, then the coefficient of the same odd degree in $P(-x)=0$ is $-a$. The coefficients of even degree terms of both $P(x)=0$ and $P(-x)=0$ are same. Thus the given condition implies that the sum of all coefficients of $P(-x)=0$ is zero and hence 1 is a root of $P(-x)=0$ which says that -1 is a root of $P(x)=0$. The rest of the problem of solving the equation is easy.

Example 3.18

Solve the equation $2x^3+11x^2-9x-18=0$.

Solution

We observe that the sum of the coefficients of the odd powers and that of the even powers are equal. Hence -1 is a root of the equation. To find other roots, we divide $2x^3+11x^2-9x-18$ by $x+1$ and get $2x^2+9x-18$ as the quotient. Solving this we get $\frac{3}{2}$ and -6 as roots. Thus $-6,-1,\frac{3}{2}$ are the roots or solutions of the given equation. ■

3.7.5 Roots in Progressions

As already noted to solve higher degree polynomial equations, we need some information about the solutions of the equation or about the polynomial. "The roots are in arithmetic progression" and "the roots are in geometric progression" are some of such information. Let us discuss an equation of this type.

Example 3.19

Obtain the condition that the roots of $x^3 + px^2 + qx + r = 0$ are in A.P.

Solution

Let the roots be in A.P. Then, we can assume them in the form $\alpha - d, \alpha, \alpha + d$.

Applying the Vieta's formula $(\alpha - d) + \alpha + (\alpha + d) = -\frac{p}{1} = p \Rightarrow 3\alpha = -p \Rightarrow \alpha = -\frac{p}{3}$.

But, we note that α is a root of the given equation. Therefore, we get

$$\left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0 \Rightarrow 9pq = 2p^3 + 27r.$$

Example 3.20

Find the condition that the roots of $ax^3 + bx^2 + cx + d = 0$ are in geometric progression. Assume $a, b, c, d \neq 0$

Solution

Let the roots be in G.P.

Then, we can assume them in the form $\frac{\alpha}{\lambda}, \alpha, \alpha\lambda$.

Applying the Vieta's formula, we get

$$\sum_1 = \alpha\left(\frac{1}{\lambda} + 1 + \lambda\right) = -\frac{b}{a} \quad \ldots (1)$$

$$\sum_2 = \alpha^2\left(\frac{1}{\lambda} + 1 + \lambda\right) = \frac{c}{a} \quad \ldots (2)$$

$$\sum_3 = \alpha^3 = -\frac{d}{a}. \quad \ldots (3)$$

Dividing (2) by (1), we get

$$\alpha = -\frac{c}{b} \quad \ldots (4)$$

Substituting (4) in (3), we get $\left(-\frac{c}{b}\right)^3 = -\frac{d}{a} \Rightarrow ac^3 = db^3$.

Example 3.21

If the roots of $x^3 + px^2 + qx + r = 0$ are in H.P., prove that $9pqr = 27r^2 + 2q^3$.

Assume $p, q, r \neq 0$

Solution

Let the roots be in H.P. Then, their reciprocals are in A.P. and roots of the equation

$$\left(\frac{1}{x}\right)^3 + p\left(\frac{1}{x}\right)^2 + q\left(\frac{1}{x}\right) + r = 0 \Leftrightarrow rx^3 + qx^2 + px + 1 = 0. \quad \ldots (1)$$

Since the roots of (1) are in A.P., we can assume them as $\alpha - d, \alpha, \alpha + d$.

Applying the Vieta's formula, we get

$$\sum_1 = (\alpha - d) + \alpha + (\alpha + d) = -\frac{q}{r} \Rightarrow 3\alpha = -\frac{q}{r} \Rightarrow \alpha = -\frac{q}{3r}.$$

But, we note that α is a root of (1). Therefore, we get

$$r\left(-\frac{q}{3r}\right)^3+q\left(-\frac{q}{3r}\right)^2+p\left(-\frac{q}{3r}\right)+1=0 \Rightarrow -q^3+3q^3-9pqr+27r^2=0 \Rightarrow 9pqr=2q^3+27r^2.$$ ■

Example 3.22

It is known that the roots of the equation $x^3-6x^2-4x+24=0$ are in arithmetic progression. Find its roots.

Solution

Let the roots be $a-d, a, a+d$. Then the sum of the roots is $3a$ which is equal to 6 from the given equation. Thus $3a=6$ and hence $a=2$. The product of the roots is a^3-ad^2 which is equal to -24 from the given equation. Substituting the value of a, we get $8-2d^2=-24$ and hence $d=\pm 4$. If we take $d=4$ we get $-2,2,6$ as roots and if we take $d=-4$, we get $6,2,-2$ as roots (same roots given in reverse order) of the equation. ■

EXERCISE 3.3

1. Solve the cubic equation : $2x^3-x^2-18x+9=0$ if sum of two of its roots vanishes.
2. Solve the equation $9x^3-36x^2+44x-16=0$ if the roots form an arithmetic progression.
3. Solve the equation $3x^3-26x^2+52x-24=0$ if its roots form a geometric progression.
4. Determine k and solve the equation $2x^3-6x^2+3x+k=0$ if one of its roots is twice the sum of the other two roots.
5. Find all zeros of the polynomial $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$, if it is known that $1+2i$ and $\sqrt{3}$ are two of its zeros.
6. Solve the cubic equations : (i) $2x^3-9x^2+10x=3$, (ii) $8x^3-2x^2-7x+3=0$.
7. Solve the equation : $x^4-14x^2+45=0$.

3.7.6 Partly Factored Polynomials

Quartic polynomial equations of the form $(ax+b)(cx+d)(px+q)(rx+s)+k=0$, $k\neq 0$ which can be rewritten in the form $(\alpha x^2+\beta x+\lambda)\ (\alpha x^2+\beta x+\mu)+k=0$

We illustrate the method of solving this situation in the next two examples.

Example 3.23

Solve the equation

$$(x-2)(x-7)(x-3)(x+2)+19 = 0.$$

Solution

We can solve this fourth degree equation by rewriting it suitably and adopting a technique of substitution. Rewriting the equation as

$$(x-2)(x-3)(x-7)(x+2)+19 = 0.$$

the given equation becomes

$$(x^2-5x+6)(x^2-5x-14)+19 = 0.$$

If we take x^2-5x as y, then the equation becomes $(y+6)(y-14)+19=0$;

that is,

$$y^2 - 8y - 65 = 0.$$

Solving this we get solutions $y = 13$ and $y = -5$. Substituting this we get two quadratic equations

$$x^2 - 5x - 13 = 0 \text{ and } x^2 - 5x + 5 = 0$$

which can be solved by usual techniques. The solutions obtained for these two equations together give solutions as $\frac{5 \pm \sqrt{77}}{2}, \frac{5 \pm \sqrt{5}}{2}$.

Example 3.24

Solve the equation $(2x-3)(6x-1)(3x-2)(x-2) - 5 = 0$.

Solution

The given equation is same as

$$(2x-3)(3x-2)(6x-1)(x-2) - 5 = 0.$$

After a computation, the above equation becomes

$$(6x^2 - 13x + 6)(6x^2 - 13x + 2) - 5 = 0.$$

By taking $y = 6x^2 - 13x$, the above equation becomes,

$$(y+6)(y+2) - 5 = 0$$

which is same as

$$y^2 + 8y + 7 = 0.$$

Solving this equation, we get $y = -1$ and $y = -7$.

Substituting the values of y in $y = 6x^2 - 13x$, we get

$$6x^2 - 13x + 1 = 0$$

$$6x^2 - 13x + 7 = 0$$

Solving these two equations, we get

$$x = 1, x = \frac{7}{6}, x = \frac{13 + \sqrt{145}}{12} \text{ and } x = \frac{13 - \sqrt{145}}{12}$$

as the roots of the given equation.

EXERCISE 3.4

1. Solve : (i) $(x-5)(x-7)(x+6)(x+4) = 504$ (ii) $(x-4)(x-7)(x-2)(x+1) = 16$
2. Solve : $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$

3.8 Polynomial Equations with no Additional Information

3.8.1 Rational Root Theorem

We can find a few roots of some polynomial equations by trial and error method. For instance, we consider the equation

$$4x^3 - 8x^2 - x + 2 = 0 \qquad \ldots (1)$$

This is a third degree equation which cannot be solved by any method so far we discussed in this chapter. If we denote the polynomial in (1) as $P(x)$, then we see that $P(2) = 0$ which says that $x - 2$ is a factor. As the rest of the problem of solving the equation is easy, we leave it as an exercise.

Example 3.25

Solve the equation $x^3-5x^2-4x+20=0$.

Solution

If $P(x)$ denotes the polynomial in the equation, then $P(2)=0$. Hence 2 is a root of the polynomial equations. To find other roots, we divide the given polynomial $x^3-5x^2-4x+20$ by $x-2$ and get $Q(x)=$ $x^2-3x-10$ as the quotient. Solving $Q(x)=0$ we get -2 and 5 as roots. Thus $2,-2,5$ are the solutions of the given equation. ■

Guessing a number as a root by trial and error method is not an easy task. But when the coefficients are integers, using its leading coefficient and the constant term, we can list certain rational numbers as possible roots. Rational Root Theorem helps us to create such a list of possible rational roots. We recall that if a polynomial has rational coefficients, then by multiplying by suitable numbers we can obtain a polynomial with integer coefficients having the same roots. So we can use **Rational Root Theorem**, given below, to guess a few roots of polynomial with rational coefficient. We state the theorem without proof.

E5WUH

Theorem 3.5 (Rational Root Theorem)

Let $a_nx^n+\cdots+a_1x+a_0$ with $a_n\neq 0$ and $a_0\neq 0$, be a polynomial with integer coefficients. If $\frac{p}{q}$, with $(p,q)=1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n.

When $a_n=1$, if there is a rational root $\dfrac{p}{q}$, then as per theorem 3.5 q is a factor of a_n, then we must have $q=\pm 1$. Thus p must be an integer. So a monic polynomial with integer coefficient cannot have non-integral rational roots. So when $a_n=1$, if at all there is a rational root, it must be an integer and the integer should divide a_0. (We say an integer a divides an integer b, if $b=ad$ for some integer d.)

As an example let us consider the equation $x^2-5x-6=0$. The divisors of 6 are $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$. From Rational Root Theorem, we can conclude that $\pm 1,\ \pm 2,\ \pm 3,\ \pm 6$ are the only possible solutions of the equation. It does not mean that all of them are solutions. The two values -1 and 6 satisfy the equation and other values do not satisfy the equation.

Moreover, if we consider the equation $x^2+4=0$, according to the Rational Root theorem, the possible solutions are $\pm 1,\ \pm 2,\ \pm 4$; but none of them is a solution. The Rational Root Theorem helps us only to guess a solution and it does not give a solution.

Example 3.26

Find the roots of $2x^3+3x^2+2x+3=0$.

Solution

According to our notations, $a_n=2$ and $a_0=3$. If $\dfrac{p}{q}$ is a zero of the polynomial, then as $(p,q)=1$, p must divide 3 and q must divide 2. Clearly, the possible values of p are $1,-1,3,-3$ and the possible values of q are $1,-1,2,-2$. Using these p and q we can form only the fractions

$\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{3}{1}$. Among these eight possibilities, after verifying by substitution, we get $\frac{-3}{2}$ is the only rational zero. To find other zeros, we divide the given polynomial $2x^3+3x^2+2x+3$ by $2x+3$ and get x^2+1 as the quotient with zero remainder. Solving $x^2+1=0$, we get i and $-i$ as roots. Thus $\frac{-3}{2}, -i, i$ are the roots of the given polynomial equation. ∎

3.8.2 Reciprocal Equations

Let α be a solution of the equation.

$$2x^6-3x^5+\sqrt{2}x^4+7x^3+\sqrt{2}x^2-3x+2 = 0. \qquad \text{... (1)}$$

Then $\alpha \neq 0$ (why?) and

$$2\alpha^6-3\alpha^5+\sqrt{2}\alpha^4+7\alpha^3+\sqrt{2}\alpha^2-3\alpha+2 = 0.$$

Substituting $\frac{1}{\alpha}$ for x in the left side of (1), we get

$$2\left(\frac{1}{\alpha}\right)^6-3\left(\frac{1}{\alpha}\right)^5+\sqrt{2}\left(\frac{1}{\alpha}\right)^4+7\left(\frac{1}{\alpha}\right)^3+\sqrt{2}\left(\frac{1}{\alpha}\right)^2-3\left(\frac{1}{\alpha}\right)+2$$

$$= \frac{2-3\alpha+\sqrt{2}\alpha^2+7\alpha^3+\sqrt{2}\alpha^4-3\alpha^5+2\alpha^6}{\alpha^6} = \frac{0}{\alpha^6} = 0.$$

Thus $\frac{1}{\alpha}$ is also a solution of (1). Similarly we can see that if α is a solution of the equation

$$2x^5+3x^4-4x^3+4x^2-3x-2 = 0 \qquad \text{... (2)}$$

then $\frac{1}{\alpha}$ is also a solution of (2).

Equations (1) and (2) have a common property that, if we replace x by $\frac{1}{x}$ in the equation and write it as a polynomial equation, then we get back the same equation. The immediate question that flares up in our mind is "Can we identify whether a given equation has this property or not just by seeing it?" Theorem 3.6 below answers this question.

Definition 3.1

A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial if one of the following conditions is true:

(i) $P(x)=x^nP\left(\frac{1}{x}\right)$ (ii) $P(x)=-x^nP\left(\frac{1}{x}\right)$.

A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial of Type I if $P(x)=x^nP\left(\frac{1}{x}\right)$. is called a reciprocal equation of Type I.

A polynomial $P(x)$ of degree n is said to be a reciprocal polynomial of Type II $P(x)=-x^nP\left(\frac{1}{x}\right)$. is called a reciprocal equation of Type II.

Theorem 3.6

A polynomial equation $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0$, $(a_n \neq 0)$ is a reciprocal equation if, and only if, one of the following two statements is true:

(i) $a_n = a_0$, $a_{n-1} = a_1$, $a_{n-2} = a_2 \cdots$

(ii) $a_n = -a_0$, $a_{n-1} = -a_1$, $a_{n-2} = -a_2, \cdots$

Proof

Consider the polynomial equation

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 = 0. \qquad \ldots (1)$$

Replacing x by $\frac{1}{x}$ in (1), we get

$$P\left(\frac{1}{x}\right) = \frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \frac{a_{n-2}}{x^{n-2}} + \cdots + \frac{a_2}{x^2} + \frac{a_1}{x} + a_0 = 0. \qquad \ldots (2)$$

Multiplying both sides of (2) by x^n, we get

$$x^nP\left(\frac{1}{x}\right) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0. \qquad \ldots (3)$$

Now, (1) is a reciprocal equation $\Leftrightarrow$ $P(x) = \pm\, x^nP\left(\frac{1}{x}\right) \Leftrightarrow$ (1) and (3) are same .

This is possible $\Leftrightarrow$ $\frac{a_n}{a_0} = \frac{a_{n-1}}{a_1} = \frac{a_{n-2}}{a_2} = \cdots = \frac{a_2}{a_{n-2}} = \frac{a_1}{a_{n-1}} = \frac{a_0}{a_n}$.

Let the proportion be equal to λ. Then, we get $\frac{a_n}{a_0} = \lambda$ and $\frac{a_0}{a_n} = \lambda$. Multiplying these equations, we get $\lambda^2 = 1$. So, we get two cases $\lambda = 1$ and $\lambda = -1$.

Case (i) :

$\lambda = 1$ In this case, we have $a_n = a_0,\ a_{n-1} = a_1,\ a_{n-2} = a_2,\ \cdots$.

That is, the coefficients of (1) from the beginning are equal to the coefficients from the end.

Case (ii) :

$\lambda = -1$ In this case, we have $a_n = -a_0,\ a_{n-1} = -a_1,\ a_{n-2} = -a_2,\ \cdots$.

That is, the coefficients of (1) from the beginning are equal in magnitude to the coefficients from the end, but opposite in sign. ■

Note

Reciprocal equations of Type I correspond to those in which the coefficients from the beginning are equal to the coefficients from the end.

For instance, the equation $6x^5 + x^4 - 43x^3 - 43x^2 + x + 6 = 0$ is of type I.

Reciprocal equations of Type II correspond to those in which the coefficients from the beginning are equal in magnitude to the coefficients from the end, but opposite in sign.

For instance, the equation $6x^5 - 41x^4 + 97x^3 - 97x^2 + 41x - 6 = 0$ is of Type II.

Remark

(i) A reciprocal equation cannot have 0 as a solution.

(ii) The coefficients and the solutions are not restricted to be real.

(iii) The statement "**If $P(x)=0$ is a polynomial equation such that whenever α is a root, $\frac{1}{\alpha}$ is also a root, then the polynomial equation $P(x)=0$ must be a reciprocal equation**" is not true. For instance $2x^3-9x^2+12x-4=0$ is a polynomial equation whose roots are $2, 2, \frac{1}{2}$. Note that $x^3P\left(\frac{1}{x}\right) \neq \pm P(x)$ and hence it is not a reciprocal equation. Reciprocal equations are classified as Type I and Type II according to $a_{n-r}=a_r$ or $a_{n-r}=-a_r$, $r=0, 1, 2, \ldots n$. We state some results without proof :

- For an odd degree reciprocal equation of Type I, $x=-1$ must be a solution.
- For an odd degree reciprocal equation of Type II, $x=1$ must be a solution.
- For an even degree reciprocal equation of Type II, the middle term must be 0. Further $x=1$ and $x=-1$ are solutions.
- For an even degree reciprocal equation, by taking $x+\frac{1}{x}$ or $x-\frac{1}{x}$ as y, we can obtain a polynomial equation of degree one half of the degree of the given equation ; solving this polynomial equation, we can get the roots of the given polynomial equation.

As an illustration, let us consider the polynomial equation

$$6x^6-35x^5+56x^4-56x^2+35x-6=0$$

which is an even degree reciprocal equation of Type II. So 1 and -1 are two solutions of the equation and hence x^2-1 is a factor of the polynomial. Dividing the polynomial by the factor x^2-1, we get $6x^4-35x^3+62x^2-35x+6$ as a factor. Dividing this factor by x^2 and rearranging the terms we get $6\left(x^2+\frac{1}{x^2}\right)-35\left(x+\frac{1}{x}\right)+62$. Setting $u=\left(x+\frac{1}{x}\right)$ it becomes a quadratic polynomial as $6(u^2-2)-35u+62$ which reduces to $6u^2-35u+50$. Solving we obtain $u=\frac{10}{3}, \frac{5}{2}$. Taking $u=\frac{10}{3}$ gives $x=3, \frac{1}{3}$ and taking $u=\frac{5}{2}$ gives $x=2, \frac{1}{2}$. So the required solutions are $+1, -1, 2, \frac{1}{2}, 3, \frac{1}{3}$.

Example 3.27

Solve the equation $7x^3-43x^2=43x-7$.

Solution

The given equation can be written as $7x^3-43x^2-43x+7=0$.

This is an odd degree reciprocal equation of Type I. Thus -1 is a solution and hence $x+1$ is a factor. Dividing the polynomial $7x^3-43x^2-43x+7$ by the factor $x+1$,we get $7x^2-50x+7$ as a quotient. Solving this we get 7 and $\frac{1}{7}$ as roots. Thus $-1, \frac{1}{7}, 7$ are the solutions of the given equation. ■

Example 3.28

Solve the following equation: $x^4-10x^3+26x^2-10x+1=0$.

Solution

This equation is Type I even degree reciprocal equation. Hence it can be rewritten as $x^2\left[\left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26\right]=0$ Since $x\neq 0$, we get $\left(x^2+\frac{1}{x^2}\right)-10\left(x+\frac{1}{x}\right)+26=0$

Let $y = x+\frac{1}{x}$. Then, we get

$\left(y^2-2\right)-10y+26 = 0 \Rightarrow y^2-10y+24 = 0 \Rightarrow (y-6)(y-4)=0 \Rightarrow y=6$ or $y=4$

Case (i)

$y=6 \quad \Rightarrow x+\frac{1}{x}=6 \Rightarrow x=3+2\sqrt{2}, x=3-2\sqrt{2}$.

Case (ii)

$y=4 \quad \Rightarrow x+\frac{1}{x}=4 \Rightarrow x=2+\sqrt{3}, x=2-\sqrt{3}$

Hence, the roots are $3\pm 2\sqrt{2},\ 2\pm\sqrt{3}$

3.8.3 Non-polynomial Equations

Some non-polynomial equations can be solved using polynomial equations. As an example let us consider the equation $\sqrt{15-2x}=x$. First we note that this is not a polynomial equation. Squaring both sides, we get $x^2+2x-15=0$. We know how to solve this polynomial equation. From the solutions of the polynomial equation, we can analyse the given equation. Clearly 3 and -5 are solutions of $x^2+2x-15=0$. If we adopt the notion of assigning only nonnegative values for $\sqrt{\bullet}$ then $x=3$ is the only solution; if we do not adopt the notion, then we get $x=-5$ is also a solution.

Example 3.29: Find solution, if any, of the equation

$$2\cos^2 x-9\cos x+4 = 0. \qquad \ldots (1)$$

Solution

The left hand side of this equation is not a polynomial in x. But it looks like a polynomial. In fact, we can say that this is a polynomial in $\cos x$. However, we can solve equation (1) by using our knowledge on polynomial equations. If we replace $\cos x$ by y, then we get the polynomial equation $2y^2-9y+4=0$ for which 4 and $\frac{1}{2}$ are solutions.

From this we conclude that x must satisfy $\cos x=4$ or $\cos x=\frac{1}{2}$. But $\cos x=4$ is never possible, if we take $\cos x=\frac{1}{2}$, then we get infinitely many real numbers x satisfying $\cos x=\frac{1}{2}$; in fact, for all $n\in\mathbb{Z}$, $x=2n\pi\pm\frac{\pi}{3}$ are solutions for the given equation (1).

If we repeat the steps by taking the equation $\cos^2 x-9\cos x+20=0$, we observe that this equation has no solution.

Remarks

We note that

- not all solutions of the derived polynomial equation give a solution for the given equation;
- there may be infinitely many solutions for non-polynomial equations though they look like polynomial equations;
- there may be no solution for such equations.
- the Fundamental Theorem of Algebra is proved only for polynomials; for non-polynomial expressions, we cannot talk about degree and hence we should not have any confusion on the Fundamental Theorem of Algebra having non-polynomial equations in mind.

EXERCISE 3.5

1. Solve the following equations

   (i) $\sin^2 x - 5\sin x + 4 = 0$ (ii) $12x^3 + 8x = 29x^2 - 4$

2. Examine for the rational roots of

   (i) $2x^3 - x^2 - 1 = 0$ (ii) $x^8 - 3x + 1 = 0$.

3. Solve : $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

4. Solve : $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$.

5. Solve the equations

   (i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$ (ii) $x^4 + 3x^3 - 3x - 1 = 0$

6. Find all real numbers satisfying $4^x - 3\left(2^{x+2}\right) + 2^5 = 0$.
7. Solve the equation $6x^4 - 5x^3 - 38x^2 - 5x + 6 = 0$ if it is known that $\frac{1}{3}$ is a solution.

3.9 Descartes Rule

In this section we discuss some bounds for the number of positive roots, number of negative roots and number of nonreal complex roots for a polynomial over $\mathbb{R}$. These bounds can be computed using a powerful tool called "Descartes Rule".

3.9.1 Statement of Descartes Rule

To discuss the rule we first introduce the concept of **change of sign in the coefficients** of a polynomial.

Consider the polynomial.

$$2x^7 - 3x^6 - 4x^5 + 5x^4 + 6x^3 - 7x + 8$$

For this polynomial, let us denote the sign of the coefficients using the symbols '$+$' and '$-$'as

$$+, -, -, +, +, -, +$$

Note that we have not put any symbol corresponding to x^2. We further note that 4 changes of sign occurred (at x^6, x^4, x^1 and x^0).

Definition 3.2

A **change of sign in the coefficients is said to occur** at the j^{th} power of x in a polynomial $P(x)$, if the coefficient of x^{j+1} and the coefficient of x^j (or) also coefficient of x^{j-1} coefficient of x^j are of different signs. (For zero coefficient we take the sign of the immediately preceding nonzero coefficient.)

From the number of sign changes, we get some information about the roots of the polynomial using **Descartes Rule**. As the proof is beyond the scope of the book, we state the theorem without proof.

Theorem 3.7 (Descartes Rule)

If p is the number of positive zeros of a polynomial $P(x)$ with real coefficients and s is the number of sign changes in coefficients of $P(x)$, then $s-p$ is a nonnegative even integer.

The theorem states that the number of positive roots of a polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(x)$. Further it says that the difference between the number of sign changes in coefficients of $P(x)$ and the number of positive roots of the polynomial $P(x)$ is even.

As a negative zero of $P(x)$ is a positive zero of $P(-x)$ we may use the theorem and conclude that **the number of negative zeros of the polynomial $P(x)$ cannot be more than the number of sign changes in coefficients of $P(-x)$ and the difference between the number of sign changes in coefficients of $P(-x)$ and the number of negative zeros of the polynomial $P(x)$ is even.**

As the multiplication of a polynomial by x^k, for some positive integer k, neither changes the number of positive zeros of the polynomial nor the number of sign changes in coefficients, we need not worry about the constant term of the polynomial. Some authors assume further that the constant term of the polynomial must be non zero.

We note that nothing is stated about 0 as a root, in Descartes rule. But from the very sight of the polynomial written in the customary form, one can say whether 0 is a root of the polynomial or not. Now let us verify Descartes rule by means of certain polynomials.

3.9.2 Attainment of bounds

3.9.2 (a) Bounds for the number of real roots

The polynomial $P(x) = (x+1)(x-1)(x-2)(x+i)(x-i)$ has the zeros $-1, 1, 2, -i, i$. The polynomial, in the customary form is $x^5 - 2x^4 - x + 2$. This polynomial $P(x)$ has 2 sign changes, namely at fourth and zeroth powers. Moreover,

$$P(-x) = -x^5 - 2x^4 + x + 2$$

has one sign change. By our Descartes rule, the number of positive zeros of the polynomial $P(x)$ cannot be more than 2; the number of negative zeros of the polynomial $P(x)$ cannot be more than 1. Clearly 1 and 2 are positive zeros, and -1 is the negative zero for the polynomial, $x^5 - 2x^4 - x + 2$, and hence the bounds 2 for positive zeros and the bound 1 for negative zeros are attained. We note that i and $-i$ are neither positive nor negative.

We know $(x+2)(x+3)(x+i)(x-i)$ is a polynomial with roots $-2, -3, -i, i$. The polynomial, say $P(x)$, in the customary form is $x^4 + 5x^3 + 7x^2 + 5x + 6$.

This polynomial $P(x)$ has no sign change and $P(-x) = x^4 - 5x^3 + 7x^2 - 5x + 6$ has 4 sign changes. By Descartes rule, the polynomial $P(x)$ cannot have more than 0 positive zeros and the number of negative zeros of the polynomial $P(x)$ cannot be more than 4.

As another example, we consider the polynomial.

$$x^n - {}^nC_1x^{n-1} + {}^nC_2x^{n-2} - {}^nC_3x^{n-3} + \cdots + (-1)^{n-1}\,{}^nC_{(n-1)}x + (-1)^n.$$

This is the expansion of $(x-1)^n$. This polynomial has n changes in coefficients and $P(-x)$ has no change of sign in coefficients. This shows that the number of positive zeros of the polynomial cannot be more than n and the number of negative zeros of the polynomial cannot be more than 0. The statement on negative zeros gives a very useful information that the polynomial has no negative zeros. But the statement on positive zeros gives no good information about the positive zeros, though there are exactly n positive zeros; in fact, it is well-known that for a polynomial of degree n, the number of zeros cannot be more than n and hence the number of positive zeros cannot be more than n.

3.9.2 (b) Bounds for the number of Imaginary (Nonreal Complex)roots

Using the Descartes rule, we can compute a lower bound for the number of imaginary roots. Let m denote the number of sign changes in coefficients of $P(x)$ of degree n; let k denote the number of sign changes in coefficients of $P(-x)$. Then there are at least $n-(m+k)$ imaginary roots for the polynomial $P(x)$. Using the other conclusion of the rule, namely, the difference between the number of roots and the corresponding sign changes is even, we can sharpen the bounds in particular cases.

Example 3.30

Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2$ has at least six imaginary roots.

Solution

Clearly there are 2 sign changes for the given polynomial $P(x)$ and hence number of positive roots of $P(x)$ cannot be more than two. Further, as $P(-x) = -9x^9 - 2x^5 - x^4 - 7x^2 + 2$, there is one sign change for $P(-x)$ and hence the number of negative roots cannot be more than one. Clearly 0 is not a root. So maximum number of real roots is 3 and hence there are atleast six imaginary roots. ■

Remark From the above discussion we note that the Descartes rule gives only upper bounds for the number of positive roots and number of negative roots; the Descartes rule neither gives the exact number of positive roots nor the exact number of negative roots. But we can find the exact number of positive, negative and nonreal roots in certain cases. Also, it does not give any method to find the roots.

Example 3.31

Discuss the nature of the roots of the following polynomials:

(i) $x^{2018}+1947x^{1950}+15x^{8}+26x^{6}+2019$ (ii) $x^{5}-19x^{4}+2x^{3}+5x^{2}+11$

Solution

Let $P(x)$ be the polynomial under consideration.

(i) The number of sign changes for $P(x)$ and $P(-x)$ are zero and hence it has no positive roots and no negative roots. Clearly zero is not a root. Thus the polynomial has no real roots and hence all roots of the polynomial are imaginary roots.

(ii) The number of sign changes for $P(x)$ and $P(-x)$ are 2 and 1 respectively. Hence it has at most two positive roots and at most one negative root.Since the difference between number of sign changes in coefficients of $P(-x)$ and the number of negative roots is even, we cannot have zero negative roots. So the number of negative roots is 1. Since the difference between number of sign changes in coefficient of $P(x)$ and the number of positive roots must be even, we must have either zero or two positive roots. But as the sum of the coefficients is zero, 1 is a root. Thus we must have two and only two positive roots. Obviously the other two roots are imaginary numbers. ■

EXERCISE 3.6

1. Discuss the maximum possible number of positive and negative roots of the polynomial equation $9x^{9}-4x^{8}+4x^{7}-3x^{6}+2x^{5}+x^{3}+7x^{2}+7x+2=0$.
2. Discuss the maximum possible number of positive and negative zeros of the polynomials $x^{2}-5x+6$ and $x^{2}-5x+16$. Also draw rough sketch of the graphs.
3. Show that the equation $x^{9}-5x^{5}+4x^{4}+2x^{2}+1=0$ has atleast 6 imaginary solutions.
4. Determine the number of positive and negative roots of the equation $x^{9}-5x^{8}-14x^{7}=0$.
5. Find the exact number of real zeros and imaginary of the polynomial $x^{9}+9x^{7}+7x^{5}+5x^{3}+3x$.

EXERCISE 3.7

Choose the correct or the most suitable answer from the given four alternatives :

1. A zero of $x^{3}+64$ is

   (1) 0 (2) 4 (3) $4i$ (4) -4

2. If f and g are polynomials of degrees m and n respectively, and if $h(x)=(f\circ g)(x)$, then the degree of h is

   (1) mn (2) $m+n$ (3) m^{n} (4) n^{m}

3. A polynomial equation in x of degree n always has

   (1) n distinct roots (2) n real roots (3) n complex roots (4) at most one root.

4. If α,β, and γ are the zeros of $x^{3}+px^{2}+qx+r$, then $\sum\frac{1}{\alpha}$ is

   (1) $-\frac{q}{r}$ (2) $-\frac{p}{r}$ (3) $\frac{q}{r}$ (4) $-\frac{q}{p}$

5. According to the rational root theorem, which number is not possible rational zero of $4x^{7}+2x^{4}-10x^{3}-5$?

   (1) -1 (2) $\frac{5}{4}$ (3) $\frac{4}{5}$ (4) 5

6. The polynomial $x^3 - kx^2 + 9x$ has three real zeros if and only if, k satisfies

(1) $|k| \le 6$ (2) $k = 0$ (3) $|k| > 6$ (4) $|k| \ge 6$

7. The number of real numbers in $[0, 2\pi]$ satisfying $\sin^4 x - 2\sin^2 x + 1$ is

(1) 2 (2) 4 (3) 1 (4) ∞

8. If $x^3 + 12x^2 + 10ax + 1999$ definitely has a positive zero, if and only if

(1) $a \ge 0$ (2) $a > 0$ (3) $a < 0$ (4) $a \le 0$

9. The polynomial $x^3 + 2x + 3$ has

(1) one negative and two imaginary zeros (2) one positive and two imaginary zeros

(3) three real zeros (4) no zeros

10. The number of positive zeros of the polynomial $\sum_{j=0}^{n} {}^nC_r (-1)^r x^r$ is

(1) 0 (2) n (3) $< n$ (4) r

SUMMARY

In this chapter we studied

- Vieta's Formula for polynomial equations of degree 2,3, and n>3.
- **The Fundamental Theorem of Algebra** : A polynomial of degree $n \ge 1$ has at least one root in $\mathbb{C}$.
- **Complex Conjugate Root Theorem** : Imaginary (nonreal complex) roots occur as conjugate pairs, if the coefficients of the polynomial are real.
- **Rational Root Theorem** : Let $a_n x^n + \cdots + a_1 x + a_0$ with $a_n \ne 0$ and $a_0 \ne 0$, be a polynomial with integer coefficients. If $\frac{p}{q}$, with $(p, q) = 1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n.
- Methods to solve some special types of polynomial equations like polynomials having only even powers, partly factored polynomials, polynomials with sum of the coefficients is zero, reciprocal equations.
- **Descartes Rule :** If p is the number of positive roots of a polynomial $P(x)$ and s is the number of sign changes in coefficients of $P(x)$, then $s - p$ is a nonnegative even integer.

ICT CORNER

https://ggbm.at/vchq92pg or Scan the QR Code

Open the Browser, type the URL Link given below (or) Scan the QR code. GeoGebra work book named "12th Standard Mathematistics" will open. In the left side of the work book there are many chapters related to your text book. Click on the chapter named "Theory of Equations". You can see several work sheets related to the chapter. Select the work sheet "Relation between roots and co-efficients"

Chapter 4

Inverse Trigonometric Functions

QBML8

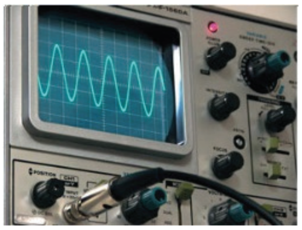
"The power of Mathematics is often to change one thing into another, to change geometry into language"

- Marcus du Sautoy

4.1 Introduction

John F.W. Herschel

In everyday life, indirect measurement is used to obtain solutions to problems that are impossible to solve using measurement tools. Trigonometry helps us to find measurements like heights of mountains and tall buildings without using measurement tools. Trigonometric functions and their inverse trigonometric functions are widely used in engineering and in other sciences including physics. They are useful not only in solving triangles, given the length of two sides of a right triangle, but also they help us in evaluating a certain type of integrals, such as $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ and $\int \frac{1}{x^2+a^2}\,dx$. The symbol $\sin^{-1} x$ denoting the inverse trigonometric function arcsine(x) of sine function was introduced by the British mathematician **John F.W.Herschel (1792-1871)**. For his work along with his father, he was presented with the Gold Medal of the Royal Astronomical Society in 1826.



An oscilloscope is an electronic device that converts electrical signals into graphs like that of sine function. By manipulating the controls, we can change the amplitude, the period and the phase shift of sine curves. The oscilloscope has many applications like measuring human heartbeats, where the trigonometric functions play a dominant role.

Let us consider some simple situations where inverse trigonometric functions are often used.

Illustration-1 (Slope problem)

Consider a straight line $y = mx + b$. Let us find the angle θ made by the line with x-axis in terms of slope m. The slope or gradient m is defined as the rate of change of a function, usually calculated by $m = \frac{\Delta y}{\Delta x}$. From right triangle (Fig. 4.1), $\tan\theta = \frac{\Delta y}{\Delta x}$.

Thus, $\tan\theta = m$. In order to solve for θ, we need the inverse trigonometric function called **"inverse tangent function"**.

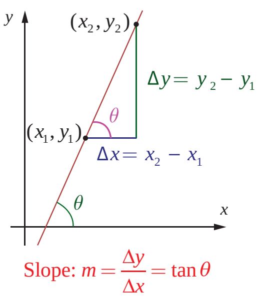


Fig. 4.1

Illustration-2 (Movie Theatre Screens)

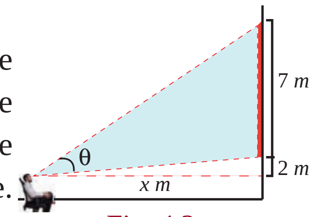


Fig. 4.2

Suppose that a movie theatre has a screen of 7 metres tall. When someone sits down, the bottom of the screen is 2 metres above the eye level. The angle formed by drawing a line from the eye to the bottom of the screen and a line from the eye to the top of the screen is called the viewing angle. In Fig. 4.2, θ is the viewing angle. Suppose that the person sits x metres away from the screen. The viewing angle θ is given by the function $\theta(x) = \tan^{-1}\left(\frac{9}{x}\right) - \tan^{-1}\left(\frac{2}{x}\right)$. Observe that the viewing angle θ is a function of x.

ZLH9D

Illustration-3 (Drawbridge)

Assume that there is a double-leaf drawbridge as shown in Fig.4.3. Each leaf of the bridge is 40 metres long. A ship of 33 metres wide needs to pass through the bridge. Inverse trigonometric function helps us to find the minimum angle θ so that each leaf of the bridge should be opened in order to ensure that the ship will pass through the bridge.

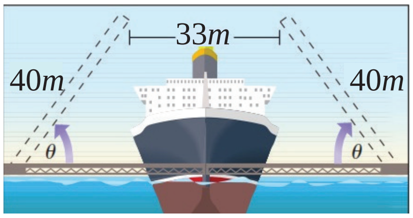


Fig. 4.3

In class XI, we have discussed trigonometric functions of real numbers using unit circle, where the angles are in radian measure. In this chapter, we shall study the inverse trigonometric functions, their graphs and properties. In our discussion, as usual $\mathbb{R}$ and $\mathbb{Z}$ stand for the set of all real numbers and all integers, respectively. Let us recall the definition of periodicity, domain and range of six trigonometric functions.

Learning Objectives

Upon completion of this chapter, students will be able to

- define inverse trigonometric functions
- evaluate the principal values of inverse trigonometric functions
- draw the graphs of trigonometric functions and their inverses
- apply the properties of inverse trigonometric functions and evaluate some expressions

4.2 Some Fundamental Concepts

Definition 4.1 (Periodicity)

A real valued function f is **periodic** if there exists a number $p > 0$ such that for all x in the domain of f, $x + p$ is in the domain of f and $f(x+p) = f(x)$.

The smallest of all such numbers, is called the **period** of the function f.

For instance, $\sin x$, $\cos x$, $\operatorname{cosec} x$, $\sec x$ and e^{ix} are periodic functions with period 2π radians, whereas $\tan x$ and $\cot x$ are periodic functions with period π radians.

Definition 4.2 (Odd and Even functions)

A real valued function f is an **even function** if for all x in the domain of f, $-x$ is also in the domain of f and $f(-x)=f(x)$.

A real valued function f is an **odd function** if for all x in the domain of f, $-x$ is also in the domain of f and $f(-x)=-f(x)$.

For instance, x^3, $\sin x$, $\operatorname{cosec} x$, $\tan x$ and $\cot x$ are all **odd functions**, whereas x^2, $\cos x$ and $\sec x$ are **even functions**.

Remark

The period of $f=g\pm h$ is $\operatorname{lcm}\{\text{period of } g, \text{period of } h\}$, whenever they exist.

For instance, the period of $y=\cos 6x+\sin 4x$ is π and that of $y=\cos x-\sin x$ is 2π.

4.2.1 Domain and Range of trigonometric functions

The **domain** and **range** of trigonometric functions are given in the following table.

Trigonometric function	$\sin x$	$\cos x$	$\tan x$	$\operatorname{cosec} x$	$\sec x$	$\cot x$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}\setminus\left\{(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}\right\}$	$\mathbb{R}\setminus\{n\pi,\ n\in\mathbb{Z}\}$	$\mathbb{R}\setminus\left\{(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}\right\}$	$\mathbb{R}\setminus\{n\pi,\ n\in\mathbb{Z}\}$
Range	$[-1, 1]$	$[-1, 1]$	$\mathbb{R}$	$\mathbb{R}\setminus(-1, 1)$	$\mathbb{R}\setminus(-1, 1)$	$\mathbb{R}$

4.2.2 Graphs of functions

Let $f:\mathbb{R}\to\mathbb{R}$ be a real valued function and $f(x)$ be the value of the function f at a point x in the domain. Then, the set of all points $(x, f(x)), x\in\mathbb{R}$ determines the graph of the function f. In general, a graph in xy-plane need not represent a function. However, if the graph passes the vertical line test (any vertical line intersects the graph, if it does, atmost at one point), then the graph represents a function. A best way to study a function is to draw its graph and analyse its properties through the graph.

Every day, we come across many phenomena like tides, day or night cycle, which involve periodicity over time. Since trigonometric functions are periodic, such phenomena can be studied through trigonometric functions. Making a visual representation of a trigonometric function, in the form of a graph, can help us to analyse the properties of phenomena involving periodicities.

To graph the trigonometric functions in the xy-plane, we use the symbol x for the independent variable representing an angle measure in radians, and y for the dependent variable. We write $y=\sin x$ to represent the sine function, and in a similar way for other trigonometric functions. In the following sections, we discuss how to draw the graphs of trigonometric functions and inverse trigonometric functions and study their properties.

4.2.3 Amplitude and Period of a graph

The amplitude is the maximum distance of the graph from the x-axis. Thus, the amplitude of a function is the height from the x-axis to its maximum or minimum. The period is the distance required for the function to complete one full cycle.

Remark

(i) The graph of a periodic function consists of repetitions of the portion of the graph on an interval of length of its period.

(ii) The graph of an odd function is symmetric with respect to the origin and the graph of an even function is symmetric about the y-axis.

4.2.4 Inverse functions

Remember that a function is a rule that, given one value, always gives back a unique value as its answer. For existence, the inverse of a function has to satisfy the above functional requirement. Let us explain this with the help of an example.

Let us consider a set of all human beings not containing identical twins. Every human being from our set, has a blood type and a DNA sequence. These are functions, where a person is the input and the output is blood type or DNA sequence. We know that many people have the same blood type but DNA sequence is unique to each individual. Can we map backwards? For instance, if you know the blood type, do you know specifically which person it came from? The answer is NO. On the other hand, if you know a DNA sequence, a unique individual from our set corresponds to the known DNA sequence. When a function is one-to-one, like the DNA example, then mapping backward is possible. The reverse mapping is called the inverse function. Roughly speaking, the inverse function undoes what the function does.

For any right triangle, given one acute angle and the length of one side, we figure out what the other angles and sides are. But, if we are given only two sides of a right triangle, we need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play.

We know that none of the trigonometric functions is one-to-one over its entire domain. For instance, given $\sin\theta = 0.5$, we have infinitely many $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}, \cdots$ satisfying the equation. Thus, given $\sin\theta$, it is not possible to recover θ uniquely. To overcome the problem of having multiple angles mapping to the same value, we will restrict our domain suitably before defining the inverse trigonometric function.

To construct the inverse of a trigonometric function, we take an interval small enough such that the function is one-to-one in the restricted interval, but the range of the function restricted to that interval is the whole range. In this chapter, we define the inverses of trigonometric functions with their restricted domains.

4.2.5 Graphs of inverse functions

Assume that f is a bijective function and f^{-1} is the inverse of f. Then, $y = f(x)$ if and only if $x = f^{-1}(y)$. Therefore, (a, b) is a point on the graph of f if and only if (b, a) is the corresponding point on the graph of f^{-1}. This suggests that graph of the inverse function f^{-1} is obtained from the graph of f by interchanging x and y axes. In other words, the graph of f^{-1} is the mirror image of the graph of f in the line $y = x$ or equivalently, the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

4.3 Sine Function and Inverse Sine Function

Let us recall that sine function is a function with $\mathbb{R}$ as its domain and $[-1, 1]$ as its range. We write $y = \sin x$ and $y = \sin^{-1} x$ or $y = \arcsin(x)$ to represent the sine function and the inverse sine function, respectively. Here, the symbol -1 is not an exponent. It denotes the inverse and does not mean the reciprocal.

We know that $\sin(x + 2\pi) = \sin x$ is true for all real numbers x. Also, $\sin(x + p)$ need not be equal to $\sin x$ for $0 < p < 2\pi$ and for all x. Hence, the period of the sine function is 2π.

4.3.1 The graph of sine function

The graph of the sine function is the graph of $y = \sin x$, where x is a real number. Since sine function is periodic with period 2π, the graph of the sine function is repeating the same pattern in each of the intervals, $\cdots$, $[-2\pi, 0]$, $[0, 2\pi]$, $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, $\ldots$. Therefore, it suffices to determine the portion of the graph for $x \in [0, 2\pi]$. Let us construct the following table to identify some known coordinate pairs for the points (x, y) on the graph of $y = \sin x$, $x \in [0, 2\pi]$.

x (in **radian**)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0

It is clear that the graph of $y = \sin x$, $0 \le x \le 2\pi$, begins at the origin. As x increases from 0 to $\frac{\pi}{2}$, the value of $y = \sin x$ increases from 0 to 1. As x increases from $\frac{\pi}{2}$ to π and then to $\frac{3\pi}{2}$, the value of y decreases from 1 to 0 and then to -1. As x increases from $\frac{3\pi}{2}$ to 2π, the value of y increases from -1 to 0. Plot the points listed in the table and connect them with a smooth curve. The portion of the graph is shown in Fig. 4.4.

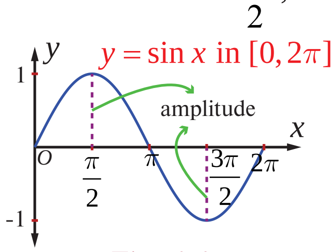


Fig. 4.4

The entire graph of $y = \sin x$, $x \in \mathbb{R}$ consists of repetitions of the above portion on either side of the interval $[0, 2\pi]$ as $y = \sin x$ is periodic with period 2π. The graph of sine function is shown in Fig. 4.5. The portion of the curve corresponding to 0 to 2π is called a **cycle**. Its amplitude is 1.

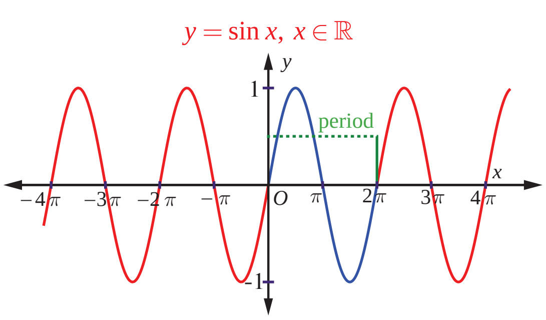


Fig. 4.5

Note

Observe that $\sin x \ge 0$ for $0 \le x \le \pi$, which corresponds to the values of the sine function in quadrants I and II and $\sin x < 0$ for $\pi < x < 2\pi$, which corresponds to the values of the sine function in quadrants III and IV.

4.3.2 Properties of the sine function

From the graph of $y=\sin x$, we observe the following properties of sine function:

(i) There is no break or discontinuities in the curve. The sine function is continuous.

(ii) The sine function is odd, since the graph is symmetric with respect to the origin.

(iii) The maximum value of sine function is 1 and occurs at $x=\cdots,-\frac{3\pi}{2},\frac{\pi}{2},\frac{5\pi}{2},\cdots$ and the minimum value is -1 and occurs at $x=\cdots,-\frac{\pi}{2},\frac{3\pi}{2},\frac{7\pi}{2},\cdots$. In otherwords, $-1\le \sin x\le 1$ for all $x\in\mathbb{R}$.

4.3.3 The inverse sine function and its properties

The sine function is not one-to-one in the entire domain $\mathbb{R}$. This is visualized from the fact that every horizontal line $y=b,\ -1\le b\le 1$, intersects the graph of $y=\sin x$ infinitely many times. In other words , the sine function does not pass the horizontal line test, which is a tool to decide the **one-to-one** status of a function. If the domain is restricted to $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then the sine function becomes one to one and onto **(bijection)** with the range $[-1,1]$. Now, let us define the inverse sine function with $[-1,1]$ as its domain and with $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ as its range.

Definition 4.3

For $-1\le x\le 1$, define $\sin^{-1}x$ as the unique number y in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ such that $\sin y=x$. In other words, the inverse sine function $\sin^{-1}:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is defined by $\sin^{-1}(x)=y$ if and only if $\sin y=x$ and $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Note

(i) The sine function is one-to-one on the restricted domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, but not on any larger interval containing the origin.

(ii) The cosine function is non-negative on the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, the range of $\sin^{-1}x$. This observation is very important for some of the trigonometric substitutions in Integral Calculus.

(iii) Whenever we talk about the inverse sine function, we have,

$$\sin:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1] \quad\text{and}\quad \sin^{-1}:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

(iv) We can also restrict the domain of the sine function to any one of the intervals, $\cdots\left[-\frac{5\pi}{2},-\frac{3\pi}{2}\right],\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right],\left[\frac{\pi}{2},\frac{3\pi}{2}\right],\left[\frac{3\pi}{2},\frac{5\pi}{2}\right]\cdots$ where it is one-to-one and its range is $[-1,1]$.

(vi) The restricted domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is called the **principal domain** of sine function and the values of $y=\sin^{-1}x,-1\le x\le 1$, are known as **principal values** of the function $y=\sin^{-1}x$.

From the definition of $y = \sin^{-1} x$, we observe the following:

(i) $y = \sin^{-1} x$ if and only if $x = \sin y$ for $-1 \le x \le 1$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(ii) $\sin\left(\sin^{-1} x\right) = x$ if $|x| \le 1$ and has no sense if $|x| > 1$.

(iii) $\sin^{-1}\left(\sin x\right) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Note that $\sin^{-1}(\sin 2\pi) = 0 \ne 2\pi$.

(iv) $\sin^{-1}\left(\sin x\right) = \pi - x$ if $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$. Note that $-\frac{\pi}{2} \le \pi - x \le \frac{\pi}{2}$.

(v) $y = \sin^{-1} x$ is an odd function.

Remark

Let us distinguish between the equations $\sin x = \frac{1}{2}$ and $x = \sin^{-1}\left(\frac{1}{2}\right)$. To solve the equation $\sin x = \frac{1}{2}$, one has to find all values of x in the interval $(-\infty, \infty)$ such that $\sin x = \frac{1}{2}$. However, to find x in $x = \sin^{-1}\left(\frac{1}{2}\right)$, one has to find the unique value x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin x = \frac{1}{2}$.

4.3.4 Graph of the inverse sine function

The inverse sine function, $\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, receives a real number x in the interval $[-1, 1]$ as input and gives a real number y in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as output. As usual, let us find some points (x, y) using the equation $y = \sin^{-1} x$ and plot them in the xy-plane. Observe that the value of y increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ as x increases from -1 to 1. By connecting these points by a smooth curve, we get the graph of $y = \sin^{-1} x$ as shown in Fig. 4.6.

x	y
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
0	0
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	$\frac{\pi}{2}$

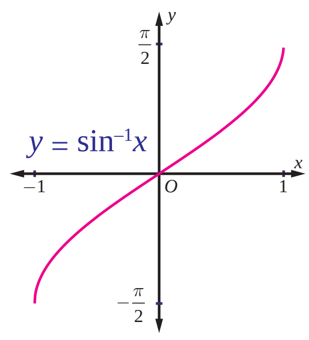


Fig. 4.6

Note

The graph of $y = \sin^{-1} x$

(i) is also obtained by reflecting the portion of the entire graph of $y = \sin x$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ about the line $y = x$ or by interchanging x and y axes from the graph of $y = \sin x$.

(ii) passes through the origin.

(iii) is symmetric with respect to the origin and hence, $y = \sin^{-1} x$ is an odd function.

We depict the graphs of both $y=\sin x,\ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$ and $y=\sin^{-1}x,\ -1\le x\le 1$ together for a better understanding.

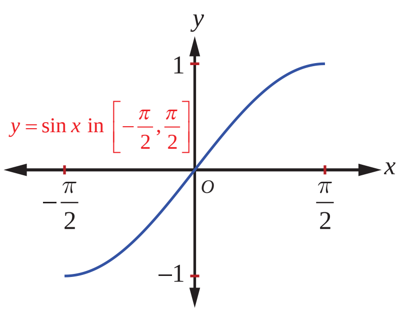


Fig. 4.7

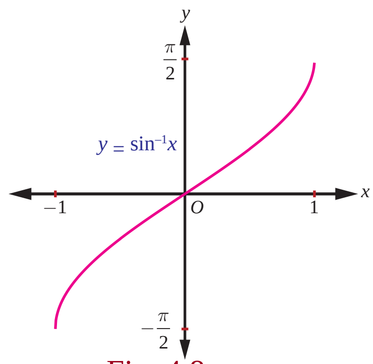


Fig. 4.8

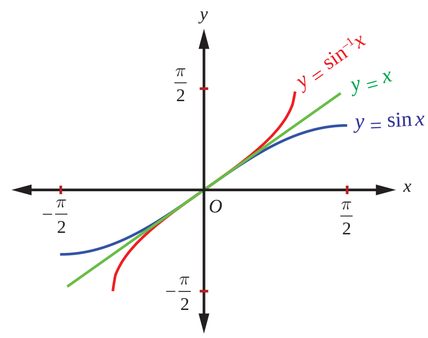


Fig. 4.9

Fig. 4.9 illustrates that the graph of $y=\sin^{-1}x$ is the mirror image of the graph of $y=\sin x,\ -\frac{\pi}{2}\le x\le\frac{\pi}{2}$, in the line $y=x$ and also shows that the sine function and the inverse sine function are symmetric with respect to the origin.

Example 4.1

Find the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ (in radians and degrees).

Solution

Let $\sin^{-1}\left(-\frac{1}{2}\right)=y$. Then $\sin y=-\frac{1}{2}$.

The **range** of the principal value of $\sin^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and hence, let us find $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ such that $\sin y=-\frac{1}{2}$. Clearly, $y=-\frac{\pi}{6}$.

Thus, the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$. This corresponds to $-30°$. ■

Example 4.2

Find the principal value of $\sin^{-1}(2)$, if it exists.

Solution

Since the domain of $y=\sin^{-1}x$ is $[-1,\ 1]$ and $2\notin[-1,1]$, $\sin^{-1}(2)$ does not exist. ■

Example 4.3

Find the principal value of

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (ii) $\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)$ (iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$.

Solution

We know that $\sin^{-1}:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is given by

$\sin^{-1}x=y$ if and only if $x=\sin y$ for $-1\le x\le 1$ and $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$. Thus,

(i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$, since $\frac{\pi}{4}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}$, since $-\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

(iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)=\sin^{-1}\left(\sin\left(\pi-\frac{\pi}{6}\right)\right)=\sin^{-1}\left(\sin\frac{\pi}{6}\right)=\frac{\pi}{6}$, since $\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. ■

Example 4.4

Find the domain of $\sin^{-1}\left(2-3x^2\right)$

Solution

We know that the domain of $\sin^{-1}(x)$ is $[-1,1]$.

This leads to $-1\le 2-3x^2\le 1$, which implies $-3\le -3x^2\le -1$.

Now, $-3\le -3x^2$, gives $x^2\le 1$ and ... (1)

$-3x^2\le -1$, gives $x^2\ge\frac{1}{3}$ ... (2)

Combining the equations (1) and (2), we get $\frac{1}{3}\le x^2\le 1$. That is, $\frac{1}{\sqrt{3}}\le |x|\le 1$, which gives

$x\in\left[-1,-\frac{1}{\sqrt{3}}\right]\cup\left[\frac{1}{\sqrt{3}},1\right]$, since $a\le|x|\le b$ implies $x\in[-b,-a]\cup[a,b]$. ■

EXERCISE 4.1

1. Find all the values of x such that

   (i) $-10\pi\le x\le 10\pi$ and $\sin x=0$ (ii) $-3\pi\le x\le 3\pi$ and $\sin x=-1$.

2. Find the period and amplitude of

   (i) $y=\sin 7x$ (ii) $y=-\sin\left(\frac{1}{3}x\right)$ (iii) $y=4\sin(-2x)$.

3. Sketch the graph of $y=\sin\left(\frac{1}{3}x\right)$ for $0\le x<6\pi$.

4. Find the value of (i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ (ii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right)$.

5. For what value of x does $\sin x=\sin^{-1}x$?

6. Find the domain of the following

   (i) $f(x)=\sin^{-1}\left(\frac{x^2+1}{2x}\right)$ (ii) $g(x)=2\sin^{-1}(2x-1)-\frac{\pi}{4}$.

7. Find the value of $\sin^{-1}\left(\sin\frac{5\pi}{9}\cos\frac{\pi}{9}+\cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$.

4.4 The Cosine Function and Inverse Cosine Function

The cosine function is a function with $\mathbb{R}$ as its domain and $[-1, 1]$ as its range. We write $y = \cos x$ and $y = \cos^{-1} x$ or $y = \arccos(x)$ to represent the cosine function and the inverse cosine function, respectively. Since $\cos(x + 2\pi) = \cos x$ is true for all real numbers x and $\cos(x + p)$ need not be equal to $\cos x$ for $0 < p < 2\pi$, $x \in \mathbb{R}$, the period of $y = \cos x$ is 2π.

4.4.1 Graph of cosine function

The graph of cosine function is the graph of $y = \cos x$, where x is a real number. Since cosine function is of period 2π, the graph of cosine function is repeating the same pattern in each of the intervals $\cdots, [-4\pi, -2\pi], [-2\pi, 0], [0, 2\pi], [2\pi, 4\pi], [4\pi, 6\pi], \cdots$. Therefore, it suffices to determine the portion of the graph of cosine function for $x \in [0, 2\pi]$. We construct the following table to identify some known coordinate pairs (x, y) for points on the graph of $y = \cos x$, $x \in [0, 2\pi]$.

x (in radian)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1

The table shows that the graph of $y = \cos x$, $0 \le x \le 2\pi$, begins at (0,1). As x increases from 0 to π, the value of $y = \cos x$ decreases from 1 to -1. As x increases from π to 2π, the value of y increases from -1 to 1. Plot the points listed in the table and connect them with a smooth curve. The portion of the graph is shown in Fig. 4.10.

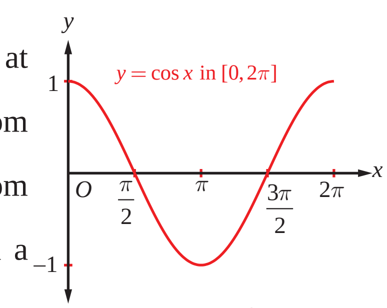


Fig. 4.10

The **graph of** $y = \cos x$, $x \in \mathbb{R}$ consists of repetitions of the above portion on either side of the interval $[0, 2\pi]$ and is shown in Fig. 4.11.

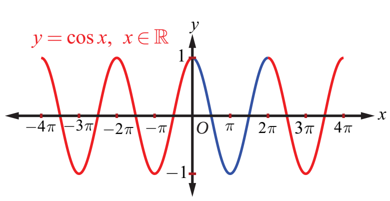


Fig. 4.11

From the graph of cosine function, observe that $\cos x$ is positive in the first quadrant $\left(\text{for } 0 \le x \le \frac{\pi}{2}\right)$, negative in the second quadrant $\left(\text{for } \frac{\pi}{2} < x \le \pi\right)$ and third quadrant $\left(\text{for } \pi < x < \frac{3\pi}{2}\right)$ and again it is positive in the fourth quadrant $\left(\text{for } \frac{3\pi}{2} < x < 2\pi\right)$.

Note

We see from the graph that $\cos(-x) = \cos x$ for all x, which asserts that $y = \cos x$ is an even function.

4.4.2 Properties of the cosine function

From the graph of $y = \cos x$, we observe the following properties of cosine function:

(i) There is no break or discontinuities in the curve. The cosine function is continuous.

(ii) The cosine function is even, since the graph is symmetric about y-axis.

(iii) The maximum value of cosine function is 1 and occurs at $x = \ldots, -2\pi, 0, 2\pi, \ldots$ and the minimum value is -1 and occurs at $x = \ldots, -\pi, \pi, 3\pi, 5\pi, \ldots$. In other words, $-1 \le \cos x \le 1$ for all $x \in \mathbb{R}$.

Remark

(i) Shifting the graph of $y = \cos x$ to the right $\frac{\pi}{2}$ radians, gives the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$, which is same as the graph of $y = \sin x$. Observe that $\cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) = \sin x$.

(ii) $y = A\sin\alpha x$ and $y = B\cos\beta x$ always satisfy the inequalities $-|A| \le A\sin\alpha x \le |A|$ and $-|B| \le B\cos\beta x \le |B|$. The amplitude and period of $y = \text{A}\sin\alpha x$ are $|A|$ and $\frac{2\pi}{|\alpha|}$, respectively and those of $y = B\cos\beta x$ are $|B|$ and $\frac{2\pi}{|\beta|}$, respectively.

The functions $y = A\sin\alpha x$ and $y = B\cos\beta x$ are known as **sinusoidal functions**.

(iii) Graphing of $y = A\sin\alpha x$ and $y = B\cos\beta x$ are obtained by extending the portion of the graphs on the intervals $\left[0, \frac{2\pi}{|\alpha|}\right]$ and $\left[0, \frac{2\pi}{|\beta|}\right]$, respectively.

Applications

Phenomena in nature like tides and yearly temperature that cycle repetitively through time are often modelled using **sinusoids**. For instance, to model tides using a general form of sinusoidal function $y = d + a\cos(bt - c)$, we give the following steps:

(i) The amplitude of a sinusoidal graph (function) is one-half of the absolute value of the difference of the maximum and minimum y-values of the graph.

Thus, **Amplitude**, $a = \frac{1}{2}(\max - \min)$; Centre line is $y = d$, where $d = \frac{1}{2}(\max + \min)$

(ii) **Period**, $p = 2 \times$ (time from maximum to minimum) ; $b = \frac{2\pi}{p}$

(iii) $c = b \times$ time at which maximum occurs.

Model-1

The depth of water at the end of a dock varies with tides. The following table shows the depth (in metres) of water at various time.

time, t	12 am	2 am	4 am	6 am	8 am	10 am	12 noon
depth	3.5	4.2	3.5	2.1	1.4	2.1	3.5

Let us construct a sinusoidal function of the form $y = d + a\cos(bt - c)$ to find the depth of water at time t. Here, $a = 1.4$; $d = 2.8$; $p = 12$; $b = \frac{\pi}{6}$; $c = \frac{\pi}{3}$.

The required sinusoidal function is $y = 2.8 + 1.4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$.

Note

The transformations of sine and cosine functions are useful in numerous applications. A circular motion is always modelled using either the sine or cosine function.

Model-2

A point rotates around a circle with centre at origin and radius 4. We can obtain the y-coordinate of the point as a function of the angle of rotation.

For a point on a circle with centre at the origin and radius a, the y-coordinate of the point is $y = a\sin\theta$, where θ is the angle of rotation. In this case, we get the equation $y(\theta) = 4\sin\theta$, where θ is in radian, the amplitude is 4 and the period is 2π . The amplitude 4 causes a vertical stretch of the y-values of the function $\sin\theta$ by a factor of 4.

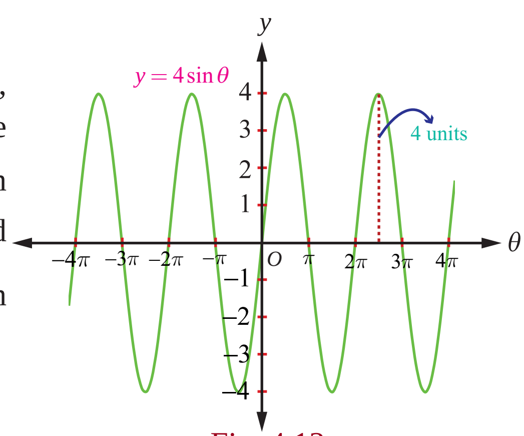


Fig. 4.12

4.4.3 The inverse cosine function and its properties

The cosine function is not one-to-one in the entire domain $\mathbb{R}$. However, the cosine function is one-to-one on the restricted domain $[0, \pi]$ and still, on this restricted domain, the range is $[-1, 1]$. Now, let us define the inverse cosine function with $[-1, 1]$ as its domain and with $[0, \pi]$ as its range.

Definition 4.4

For $-1 \le x \le 1$, define $\cos^{-1}x$ as the unique number y in $[0, \pi]$ such that $\cos y = x$. In other words, the inverse cosine function $\cos^{-1}:[-1, 1] \to [0, \pi]$ is defined by $\cos^{-1}(x) = y$ if and only if $\cos y = x$ and $y \in [0, \pi]$.

Note

(i) The sine function is non-negative on the interval $[0, \pi]$, the range of $\cos^{-1}x$. This observation is very important for some of the trigonometric substitutions in Integral Calculus.

(ii) Whenever we talk about the inverse cosine function, we have $\cos x:[0, \pi] \to [-1, 1]$ and $\cos^{-1}x:[-1, 1] \to [0, \pi]$.

(iii) We can also restrict the domain of the cosine function to any one of the intervals $\cdots, [-\pi, 0], [\pi, 2\pi], \cdots$, where it is one-to-one and its range is $[-1, 1]$.

The restricted domain $[0, \pi]$ is called the **principal domain** of cosine function and the values of $y = \cos^{-1}x, -1 \le x \le 1$, are known as **principal values** of the function $y = \cos^{-1}x$.

From the definition of $y=\cos^{-1}x$, we observe the following:

(i) $y=\cos^{-1}x$ if and only if $x=\cos y$ for $-1\le x\le 1$ and $0\le y\le\pi$.

(ii) $\cos\left(\cos^{-1}x\right)=x$ if $|x|\le 1$ and has no sense if $|x|>1$.

(iii) $\cos^{-1}\left(\cos x\right)=x$ if $0\le x\le\pi$, the range of $\cos^{-1}x$. Note that $\cos^{-1}\left(\cos 3\pi\right)=\pi$.

4.4.4 Graph of the inverse cosine function

The inverse cosine function $\cos^{-1}:[-1,1]\to[0,\pi]$, receives a real number x in the interval $[-1,1]$ as an input and gives a real number y in the interval $[0,\pi]$ as an output (an angle in radian measure). Let us find some points (x,y) using the equation $y=\cos^{-1}x$ and plot them in the xy-plane. Note that the values of y decrease from π to 0 as x increases from -1 to 1. The inverse cosine function is decreasing and continuous in the domain. By connecting the points by a smooth curve, we get the graph of $y=\cos^{-1}x$ as shown in Fig. 4.14

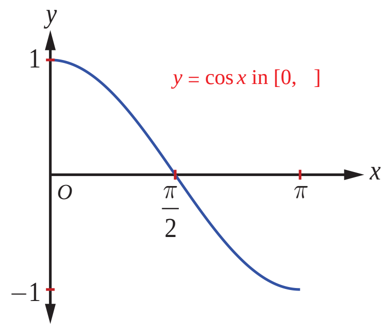


Fig. 4.13

x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0

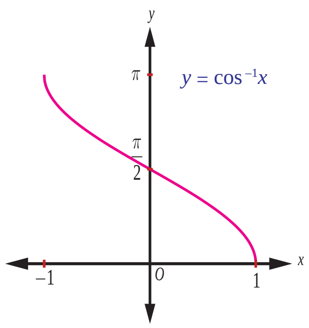


Fig. 4.14

Note

(i) The graph of the function $y=\cos^{-1}x$ is also obtained from the graph $y=\cos x$ by interchanging x and y axes.

(ii) For the function $y=\cos^{-1}x$, the x-intercept is 1 and the y-intercept is $\frac{\pi}{2}$.

(iii) The graph is not symmetric with respect to either origin or y-axis. So, $y=\cos^{-1}x$ is neither even nor odd function.

Example 4.5

Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Solution

Let $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=y$. Then, $\cos y=\frac{\sqrt{3}}{2}$.

The **range of the principal values** of $y=\cos^{-1}x$ is $[0,\pi]$.

So, let us find y in $[0,\pi]$ such that $\cos y=\frac{\sqrt{3}}{2}$.

But, $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ and $\frac{\pi}{6}\in[0,\pi]$. Therefore, $y=\frac{\pi}{6}$

Thus, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

Example 4.6

Find (i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ (iii) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right)$

Solution

It is known that $\cos^{-1} x : [-1, 1] \to [0, \pi]$ is given by

$\cos^{-1} x = y$ if and only if $x = \cos y$ for $-1 \le x \le 1$ and $0 \le y \le \pi$.

Thus, we have

(i) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$, since $\frac{3\pi}{4} \in [0, \pi]$ and $\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$.

(ii) $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$, since $-\frac{\pi}{3} \notin [0, \pi]$, but $\frac{\pi}{3} \in [0, \pi]$.

(iii) $\cos^{-1}\left(\cos\left(\frac{7\pi}{6}\right)\right) = \frac{5\pi}{6}$, since $\cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} = \cos\left(\frac{5\pi}{6}\right)$ and $\frac{5\pi}{6} \in [0, \pi]$.

Example 4.7

Find the domain of $\cos^{-1}\left(\frac{2 + \sin x}{3}\right)$.

Solution

By definition, the domain of $y = \cos^{-1} x$ is $-1 \le x \le 1$ or $|x| \le 1$. This leads to

$-1 \le \frac{2 + \sin x}{3} \le 1$ which is same as $-3 \le 2 + \sin x \le 3$.

So, $-5 \le \sin x \le 1$ reduces to $-1 \le \sin x \le 1$, which gives

$-\sin^{-1}(1) \le x \le \sin^{-1}(1)$ or $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Thus, the domain of $\cos^{-1}\left(\frac{2 + \sin x}{3}\right)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXERCISE 4.2

1. Find all values of x such that

   (i) $-6\pi \le x \le 6\pi$ and $\cos x = 0$ (ii) $-5\pi \le x \le 5\pi$ and $\cos x = 1$.

2. State the reason for $\cos^{-1}\left[\cos\left(-\frac{\pi}{6}\right)\right] \ne -\frac{\pi}{6}$.

3. Is $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ true? Justify your answer.

4. Find the principal value of $\cos^{-1}\left(\frac{1}{2}\right)$.

5. Find the value of

   (i) $2\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}(-1)$

   (iii) $\cos^{-1}\left(\cos\frac{\pi}{7}\cos\frac{\pi}{17} - \sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$.

6. Find the domain of (i) $f(x)=\sin^{-1}\left(\frac{|x|-2}{3}\right)+\cos^{-1}\left(\frac{1-|x|}{4}\right)$ (ii) $g(x)=\sin^{-1}x+\cos^{-1}x$

7. For what value of x, the inequality $\frac{\pi}{2}<\cos^{-1}(3x-1)<\pi$ holds?

8. Find the value of

(i) $\cos\left(\cos^{-1}\left(\frac{4}{5}\right)+\sin^{-1}\left(\frac{4}{5}\right)\right)$ (ii) $\cos^{-1}\left(\cos\left(\frac{4\pi}{3}\right)\right)+\cos^{-1}\left(\cos\left(\frac{5\pi}{4}\right)\right)$.

4.5 The Tangent Function and the Inverse Tangent Function

We know that the tangent function $y=\tan x$ is used to find heights or distances, such as the height of a building, mountain, or flagpole. The domain of $y=\tan x=\frac{\sin x}{\cos x}$ does not include values of x, which make the denominator zero. So, the tangent function is not defined at $x=\cdots,-\frac{3\pi}{2},-\frac{\pi}{2},\frac{\pi}{2},\frac{3\pi}{2},\cdots$. Thus, the domain of the tangent function $y=\tan x$ is $\left\{x:x\in\mathbb{R},x\neq\frac{\pi}{2}+k\pi,k\in\mathbb{Z}\right\}=\bigcup\limits_{k=-\infty}^{\infty}\left(\frac{2k+1}{2}\pi,\frac{2k+3}{2}\pi\right)$ and the range is $(-\infty,\infty)$.The tangent function $y=\tan x$ has period π.

4.5.1 The graph of tangent function

Graph of the tangent function is useful to find the values of the function over the repeated period of intervals. The tangent function is odd and hence the graph of $y=\tan x$ is symmetric with respect to the origin. Since the period of tangent function is π, we need to determine the graph over some interval of length π. Let us consider the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and construct the following table to draw the graph of $y=\tan x$, $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

x (in radian)	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$y=\tan x$	$-\sqrt{3}$	-1	$\frac{-\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Now, plot the points and connect them with a smooth curve for a partial graph of $y=\tan x$, where $-\frac{\pi}{2}\leq x\leq\frac{\pi}{2}$. If x is close to $\frac{\pi}{2}$ but remains less than $\frac{\pi}{2}$, the $\sin x$ will be close to 1 and $\cos x$ will be positive and close to 0. So, as x approaches to $\frac{\pi}{2}$, the ratio $\frac{\sin x}{\cos x}$ is positive and large and thus approaching to ∞.

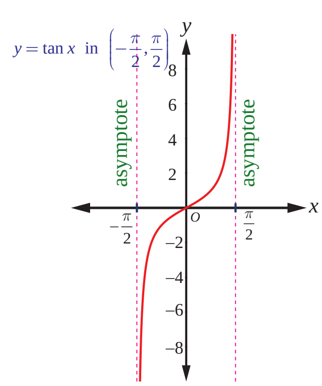


Fig. 4.15

Therefore, the line $x=\frac{\pi}{2}$ is a vertical asymptote to the graph. Similarly, if x is approaching to $-\frac{\pi}{2}$, the ratio $\frac{\sin x}{\cos x}$ is negative and large in magnitude and thus, approaching to $-\infty$. So, the line $x=-\frac{\pi}{2}$ is also a vertical asymptote to the graph. Hence, we get a branch of the graph of $y=\tan x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$ as shown in the Fig 4.15. The interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is called the **principal domain** of $y=\tan x$.

Since the tangent function is defined for all real numbers except at $x=(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}$, and is increasing, we have vertical asymptotes $x=(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}$. As branches of $y=\tan x$ are symmetric with respect to $x=n\pi, n\in\mathbb{Z}$, the entire graph of $y=\tan x$ is shown in Fig. 4.16.

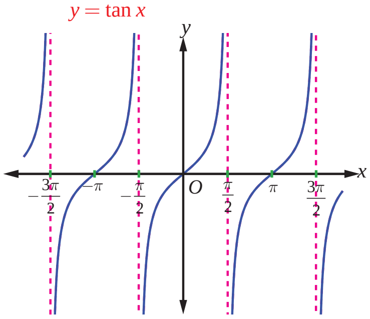


Fig. 4.16

Note

From the graph, it is seen that $y=\tan x$ is positive for $0<x<\frac{\pi}{2}$ and $\pi<x<\frac{3\pi}{2}$; $y=\tan x$ is negative for $\frac{\pi}{2}<x<\pi$ and $\frac{3\pi}{2}<x<2\pi$.

4.5.2 Properties of the tangent function

From the graph of $y=\tan x$, we observe the following properties of tangent function.

(i) The graph is not continuous and has discontinuity points at $x=(2n+1)\frac{\pi}{2},\ n\in\mathbb{Z}$.

(ii) The partial graph is symmetric about the origin for $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

(iii) It has infinitely many vertical asymptotes $x=(2n+1)\frac{\pi}{2},\ n\in\mathbb{Z}$.

(iv) The tangent function has neither maximum nor minimum.

Remark

(i) The graph of $y=a\tan bx$ goes through one complete cycle for

$$-\frac{\pi}{2|b|}<x<\frac{\pi}{2|b|} \text{ and its period is } \frac{\pi}{|b|}.$$

(ii) For $y=a\tan bx$, the asymptotes are the lines $x=\frac{\pi}{2|b|}+\frac{\pi}{|b|}k,\ k\in\mathbb{Z}$.

(iii) Since the tangent function has no maximum and no minimum value, the term **amplitude** for $\tan x$ cannot be defined.

4.5.3 The inverse tangent function and its properties

The tangent function is not one-to-one in the entire domain $\mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi, k\in\mathbb{Z}\right\}$. However, $\tan x:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$ is a bijective function. Now, we define the inverse tangent function with $\mathbb{R}$ as its domain and $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ as its range.

Definition 4.5

For any real number x, define $\tan^{-1}x$ as the unique number y in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ such that $\tan y = x$.

In other words, the inverse tangent function $\tan^{-1}:(-\infty,\infty)\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is defined by $\tan^{-1}(x)=y$ if and only if $\tan y = x$ and $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

From the definition of $y=\tan^{-1}x$, we observe the following:

(i) $y=\tan^{-1}x$ if and only if $x=\tan y$ for $x\in\mathbb{R}$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$.

(ii) $\tan\left(\tan^{-1}x\right)=x$ for any real number x and $y=\tan^{-1}x$ is an odd function.

(iii) $\tan^{-1}\left(\tan x\right)=x$ if and only if $-\frac{\pi}{2}<x<\frac{\pi}{2}$. Note that $\tan^{-1}\left(\tan\pi\right)=0$ and not π.

Note

(i) Whenever we talk about inverse tangent function, we have,

$$\tan:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R} \quad\text{and}\quad \tan^{-1}:\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right).$$

(ii) The restricted domain $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is called the **principal domain** of tangent function and the values of $y=\tan^{-1}x$, $x\in\mathbb{R}$, are known as **principal values** of the function $y=\tan^{-1}x$.

4.5.4 Graph of the inverse tangent function

$y=\tan^{-1}x$ is a function with the entire real line $(-\infty,\infty)$ as its domain and whose range is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Note that the tangent function is undefined at $-\frac{\pi}{2}$ and at $\frac{\pi}{2}$. So, the graph of $y=\tan^{-1}x$ lies strictly between the two lines $y=-\frac{\pi}{2}$ and $y=\frac{\pi}{2}$, and never touches these two lines. In other words, the two lines $y=-\frac{\pi}{2}$ and $y=\frac{\pi}{2}$ are horizontal asymptotes to $y=\tan^{-1}x$.

Fig. 4.17 and Fig. 4.18 show the graphs of $y=\tan x$ in the domain $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and $y=\tan^{-1}x$ in the domain $(-\infty,\infty)$, respectively.

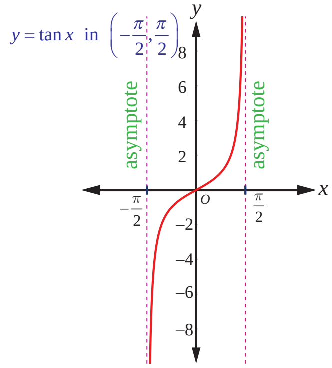


Fig. 4.17

x	y
-1	$-\frac{\pi}{4}$
0	0
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$
1	$\frac{\pi}{4}$

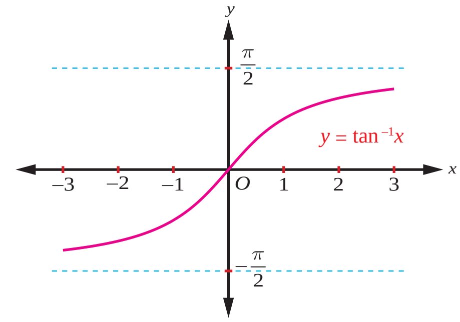


Fig. 4.18

Note

(i) The inverse tangent function is strictly increasing and continuous on the domain $(-\infty, \infty)$.

(ii) The graph of $y = \tan^{-1} x$ passes through the origin.

(iii) The graph is symmetric with respect to origin and hence, $y = \tan^{-1} x$ is an odd function.

Example 4.8

Find the principal value of $\tan^{-1}\left(\sqrt{3}\right)$.

Solution

Let $\tan^{-1}\left(\sqrt{3}\right) = y$. Then, $\tan y = \sqrt{3}$. Thus, $y = \frac{\pi}{3}$. Since $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Thus, the principal value of $\tan^{-1}\left(\sqrt{3}\right)$ is $\frac{\pi}{3}$.

Example 4.9

Find (i) $\tan^{-1}\left(-\sqrt{3}\right)$ (ii) $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$ (iii) $\tan\left(\tan^{-1}(2019)\right)$

Solution

(i) $\tan^{-1}\left(-\sqrt{3}\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$, since $-\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(ii) $\tan^{-1}\left(\tan\frac{3\pi}{5}\right)$.

Let us find $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan\theta = \tan\frac{3\pi}{5}$.

Since the tangent function has period π, $\tan\frac{3\pi}{5} = \tan\left(\frac{3\pi}{5} - \pi\right) = \tan\left(-\frac{2\pi}{5}\right)$.

Therefore, $\tan^{-1}\left(\tan\frac{3\pi}{5}\right) = \tan^{-1}\left(\tan\left(-\frac{2\pi}{5}\right)\right) = -\frac{2\pi}{5}$, since $-\frac{2\pi}{5} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(iii) Since $\tan\left(\tan^{-1} x\right) = x, x \in \mathbb{R}$, we have $\tan\left(\tan^{-1}(2019)\right) = 2019$.

Example 4.10

Find the value of $\tan^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution

Let $\tan^{-1}(-1)=y$. Then, $\tan y=-1=-\tan\frac{\pi}{4}=\tan\left(-\frac{\pi}{4}\right)$.

As $-\frac{\pi}{4}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $\tan^{-1}(-1)=-\frac{\pi}{4}$.

Now, $\cos^{-1}\left(\frac{1}{2}\right)=y$ implies $\cos y=\frac{1}{2}=\cos\frac{\pi}{3}$.

As $\frac{\pi}{3}\in[0,\pi]$, $\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$.

Now, $\sin^{-1}\left(-\frac{1}{2}\right)=y$ implies $\sin y=-\frac{1}{2}=\sin\left(-\frac{\pi}{6}\right)$.

As $-\frac{\pi}{6}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$.

Therefore, $\tan^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{4}+\frac{\pi}{3}-\frac{\pi}{6}=-\frac{\pi}{12}$.

Example 4.11

Prove that $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}}$, $-1<x<1$.

Solution

If $x=0$, then both sides are equal to 0. ... (1)

Assume that $0<x<1$.

Let $\theta=\sin^{-1}x$. Then $0<\theta<\frac{\pi}{2}$. Now, $\sin\theta=\frac{x}{1}$ gives $\tan\theta=\frac{x}{\sqrt{1-x^2}}$.

Hence, $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}}$. ... (2)

Assume that $-1<x<0$. Then, $\theta=\sin^{-1}x$ gives $-\frac{\pi}{2}<\theta<0$. Now, $\sin\theta=\frac{x}{1}$ gives $\tan\theta=\frac{x}{\sqrt{1-x^2}}$.

In this case also, $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}}$. ... (3)

Equations (1), (2) and (3) establish that $\tan\left(\sin^{-1}x\right)=\frac{x}{\sqrt{1-x^2}}$, $-1<x<1$.

EXERCISE 4.3

1. Find the domain of the following functions :

   (i) $\tan^{-1}\left(\sqrt{9-x^2}\right)$ (ii) $\frac{1}{2}\tan^{-1}\left(1-x^2\right)-\frac{\pi}{4}$.

2. Find the value of (i) $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ (ii) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$.

3. Find the value of

(i) $\tan\left(\tan^{-1}\left(\frac{7\pi}{4}\right)\right)$ (ii) $\tan\left(\tan^{-1}(1947)\right)$ (iii) $\tan\left(\tan^{-1}(-0.2021)\right)$.

4. Find the value of (i) $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)\right)$ (ii) $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{4}{5}\right)\right)$.

(iii) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right)$.

4.6 The Cosecant Function and the Inverse Cosecant Function

Like sine function, the cosecant function is an odd function and has period 2π. The values of cosecant function $y=\operatorname{cosec} x$ repeat after an interval of length 2π. Observe that $y=\operatorname{cosec} x=\frac{1}{\sin x}$ is not defined when $\sin x=0$. So, the **domain** of cosecant function is $\mathbb{R}\setminus\{n\pi : n\in\mathbb{Z}\}$. Since $-1\le\sin x\le 1$, $y=\operatorname{cosec} x$ does not take any value in between -1 and 1. Thus, the **range** of cosecant function is $(-\infty,1]\cup[1,\infty)$.

4.6.1 Graph of the cosecant function

In the interval $(0,2\pi)$, the cosecant function is continuous everywhere except at the point $x=\pi$. It has neither maximum nor minimum. Roughly speaking, the value of $y=\operatorname{cosec} x$ falls from ∞ to 1 for $x\in\left(0,\frac{\pi}{2}\right]$, it raises from 1 to ∞ for $x\in\left[\frac{\pi}{2},\pi\right)$. Again, it raises from $-\infty$ to -1 for $x\in\left(\pi,\frac{3\pi}{2}\right]$ and falls from -1 to $-\infty$ for $x\in\left[\frac{3\pi}{2},2\pi\right)$.

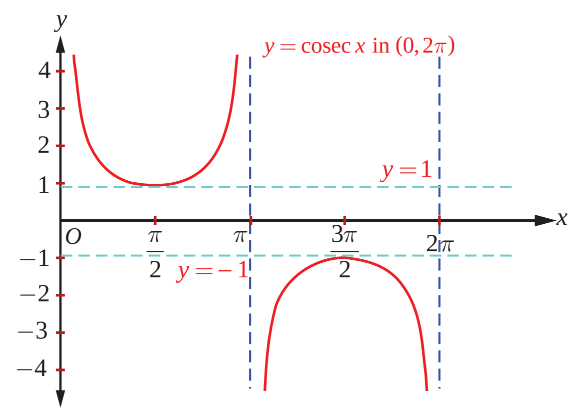


Fig. 4.19

The graph of $y=\operatorname{cosec} x$, $x\in(0,2\pi)\setminus\{\pi\}$ is shown in the Fig. 4.19. This portion of the graph is repeated for the intervals $\cdots,(-4\pi,-2\pi)\setminus\{-3\pi\}$, $(-2\pi,0)\setminus\{-\pi\},(2\pi,4\pi)\setminus\{3\pi\},(4\pi,6\pi)\setminus\{5\pi\},\cdots$.

The entire graph of $y=\operatorname{cosec} x$ is shown in Fig. 4.20.

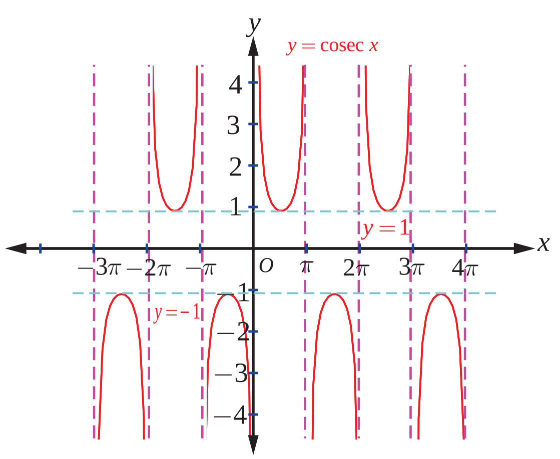


Fig. 4.20

4.6.2 The inverse cosecant function

The cosecant function, cosec : $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]\rightarrow(-\infty,-1]\ \cup\ [1,\infty)$ is bijective in the restricted domain $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$. So, the inverse cosecant function is defined with the domain $(-\infty,-1]\cup\ [1,\infty)$ and the range $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$.

Definition 4.6

The inverse cosecant function $\text{cosec}^{-1}:(-\infty,-1]\ \cup\ [1,\infty)\rightarrow\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ is defined by $\text{cosec}^{-1}(x)=y$ if and only if $\text{cosec}\, y=x$ and $y\in\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$.

4.6.3 Graph of the inverse cosecant function

The inverse cosecant function, $y=\text{cosec}^{-1}x$ is a function whose domain is $\mathbb{R}\setminus(-1,1)$ and the range is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$. That is, $\text{cosec}^{-1}:(-\infty,-1]\ \cup\ [1,\infty)\rightarrow\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$.

Fig. 4.21 and Fig. 4.22 show the graphs of cosecant function in the principal domain and the inverse cosecant function in the corresponding domain respectively.

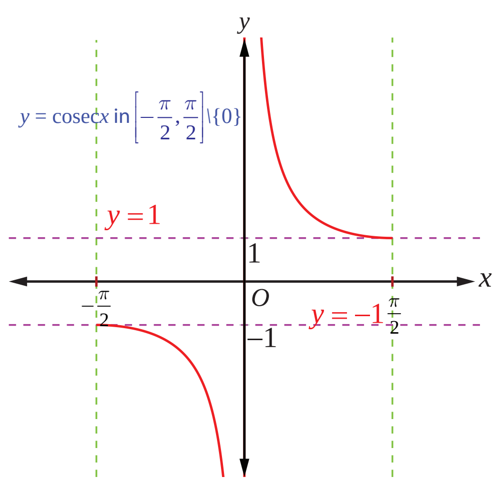


Fig. 4.21

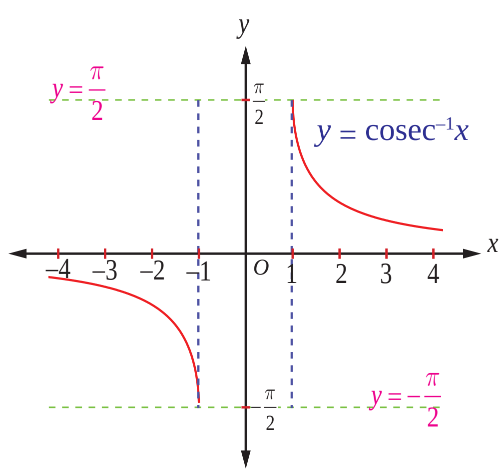


Fig. 4.22

4.7 The Secant Function and Inverse Secant Function

The secant function is defined as the reciprocal of cosine function. So, $y=\sec x=\frac{1}{\cos x}$ is defined for all values of x except when $\cos x=0$ .Thus, the **domain** of the function $y=\sec x$ is $\mathbb{R}\setminus\left\{(2n+1)\frac{\pi}{2}:n\in\mathbb{Z}\right\}$. As $-1\le\cos x\le1$, $y=\sec x$ does not take values in $(-1,\ 1)$. Thus, the range of the secant function is $(-\infty,1]\cup[1,\infty)$. The secant function has neither maximum nor minimum. The function $y=\sec x$ is a periodic function with period 2π and it is also an even function.

4.7.1 The graph of the secant function

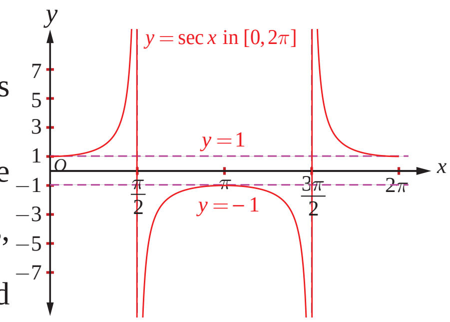


Fig. 4.23

The graph of secant function in $0 \le x \le 2\pi$, $x \ne \frac{\pi}{2}, \frac{3\pi}{2}$, is shown in Fig. 4.23. In the first and fourth quadrants or in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $y = \sec x$ takes only positive values, whereas it takes only negative values in the second and third quadrants or in the interval $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

For $0 \le x \le 2\pi$, $x \ne \frac{\pi}{2}, \frac{3\pi}{2}$, the secant function is continuous. The value of secant function raises from 1 to ∞ for $x \in \left[0, \frac{\pi}{2}\right)$; it raises from $-\infty$ to -1 for $x \in \left(\frac{\pi}{2}, \pi\right]$. It falls from -1 to $-\infty$ for $x \in \left[\pi, \frac{3\pi}{2}\right)$ and falls from ∞ to 1 for $x \in \left(\frac{3\pi}{2}, 2\pi\right]$.

As $y = \sec x$ is periodic with period 2π, the same segment of the graph for $0 \le x \le 2\pi$, $x \ne \frac{\pi}{2}, \frac{3\pi}{2}$, is repeated in $[2\pi, 4\pi] \setminus \left\{\frac{5\pi}{2}, \frac{7\pi}{2}\right\}$, $[4\pi, 6\pi] \setminus \left\{\frac{9\pi}{2}, \frac{11\pi}{2}\right\}$, $\cdots$ and in $\cdots$, $[-4\pi, -2\pi] \setminus \left\{-\frac{7\pi}{2}, -\frac{5\pi}{2}\right\}$, $[-2\pi, 0] \setminus \left\{-\frac{3\pi}{2}, -\frac{\pi}{2}\right\}$.

Now, the entire graph of $y = \sec x$ is shown in Fig. 4.24.

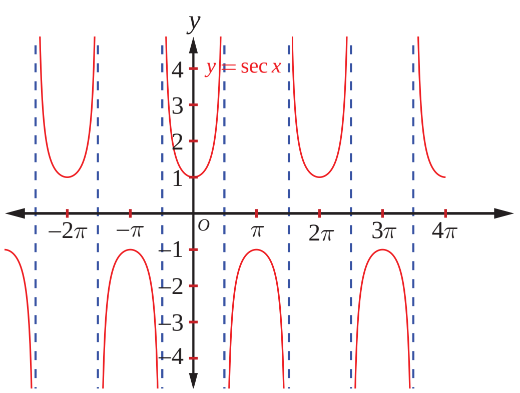


Fig. 4.24

4.7.2 Inverse secant function

The secant function, $\sec x : [0, \pi] \setminus \left\{\frac{\pi}{2}\right\} \to \mathbb{R} \setminus (-1, 1)$ is bijective in the restricted domain $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$. So, the inverse secant function is defined with $\mathbb{R} \setminus (-1, 1)$ as its domain and with $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$ as its range.

Definition 4.7

The inverse secant function $\sec^{-1} : \mathbb{R} \setminus (-1, 1) \to [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$ is defined by $\sec^{-1}(x) = y$ whenever $\sec y = x$ and $y \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$.

4.7.3 Graph of the inverse secant function

The inverse secant function, $y = \sec^{-1} x$ is a function whose domain is $\mathbb{R} \setminus (-1,1)$ and the range is $[0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$. That is, $\sec^{-1} : \mathbb{R} \setminus (-1,1) \to [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$.

Fig. 4.25 and Fig. 4.26 are the graphs of the secant function in the principal domain and the inverse secant function in the corresponding domain, respectively.

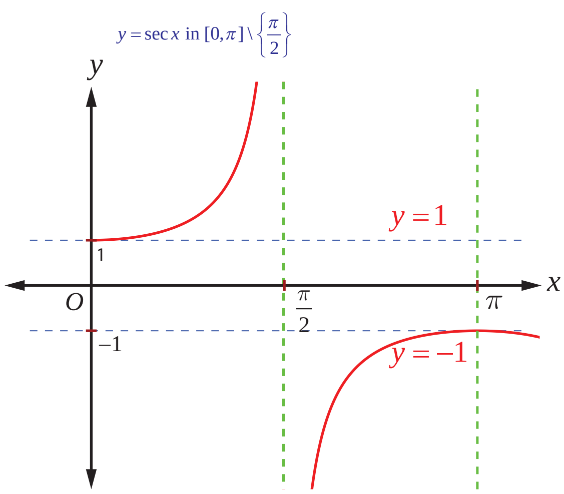


Fig. 4.25

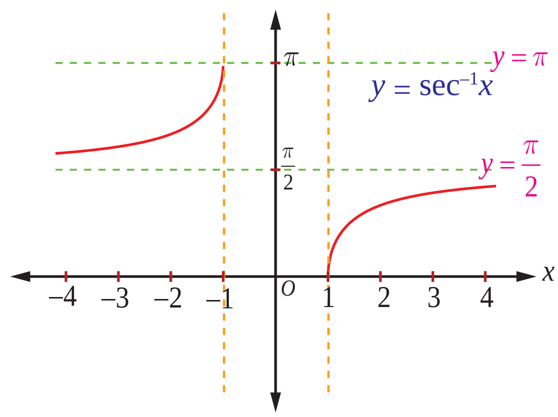


Fig. 4.26

Remark

A nice way to draw the graph of $y = \sec x$ or cosec x:

(i) Draw the graph of $y = \cos x$ or $\sin x$

(ii) Draw the vertical asymptotes at the x-intercepts and take reciprocals of y values.

4.8 The Cotangent Function and the Inverse Cotangent Function

The cotangent function is given by $\cot x = \frac{1}{\tan x}$. It is defined for all real values of x, except when $\tan x = 0$ or $x = n\pi, n \in \mathbb{Z}$. Thus, the domain of cotangent function is $\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}$ and its range is $(-\infty, \infty)$. Like $\tan x$, the cotangent function is an odd function and periodic with period π.

4.8.1 The graph of the cotangent function

The cotangent function is continuous on the set $(0, 2\pi) \setminus \{\pi\}$. Let us first draw the graph of cotangent function in $(0, 2\pi) \setminus \{\pi\}$. In the first and third quadrants, the cotangent function takes only positive values and in the second and fourth quadrants, it takes only negative values. The cotangent function has no maximum value and no minimum value. The cotangent function falls from ∞ to 0 for $x \in \left(0, \frac{\pi}{2}\right]$; falls from 0 to $-\infty$ for $x \in \left[\frac{\pi}{2}, \pi\right)$; falls from ∞ to 0 for $x \in \left(\pi, \frac{3\pi}{2}\right]$ and falls from 0 to $-\infty$ for $x \in \left[\frac{3\pi}{2}, 2\pi\right)$.

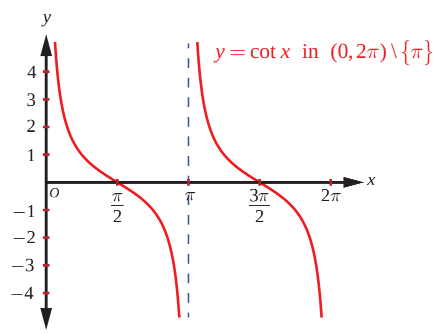


Fig. 4.27

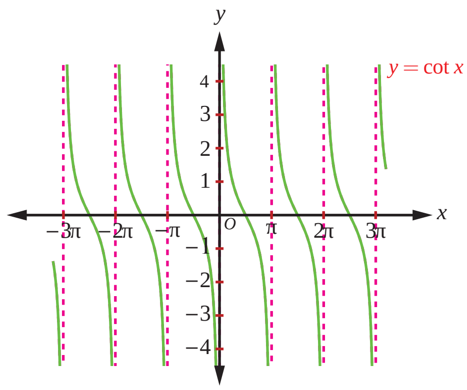


Fig. 4.28

The graph of $y = \cot x$, $x \in (0, 2\pi)\setminus\{\pi\}$ is shown in Fig 4.27. The same segment of the graph of cotangent for $(0, 2\pi)\setminus\{\pi\}$ is repeated for $(2\pi, 4\pi)\setminus\{3\pi\}, (4\pi, 6\pi)\setminus\{5\pi\}, \cdots$, and for $\cdots$, $(-4\pi, -2\pi)\setminus\{-3\pi\}$, $(-2\pi, 0)\setminus\{-\pi\}$. The entire graph of cotangent function with domain $\mathbb{R}\setminus\{n\pi : n \in \mathbb{Z}\}$ is shown in Fig. 4.28.

4.8.2 Inverse cotangent function

The cotangent function is not one-to-one in its entire domain $\mathbb{R}\setminus\{n\pi : n \in \mathbb{Z}\}$. However, $\cot : (0, \pi) \to (-\infty, \infty)$ is bijective with the restricted domain $(0, \pi)$. So, we can define the inverse cotangent function with $(-\infty, \infty)$ as its domain and $(0, \pi)$ as its range.

Definition 4.8

The inverse cotangent function $\cot^{-1} : (-\infty, \infty) \to (0, \pi)$ is defined by $\cot^{-1}(x) = y$ if and only if $\cot y = x$ and $y \in (0, \pi)$.

4.8.3 Graph of the inverse cotangent function

The inverse cotangent function, $y = \cot^{-1} x$ is a function whose domain is $\mathbb{R}$ and the range is $(0, \pi)$. That is, $\cot^{-1} x : (-\infty, \infty) \to (0, \pi)$.

Fig. 4.29 and Fig. 4.30 show the cotangent function in the principal domain and the inverse cotangent function in the corresponding domain, respectively.

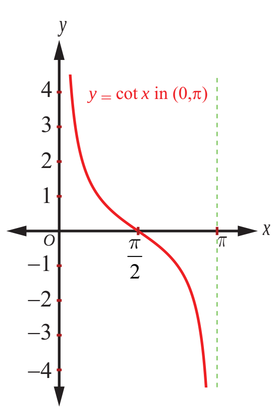


Fig. 4.29

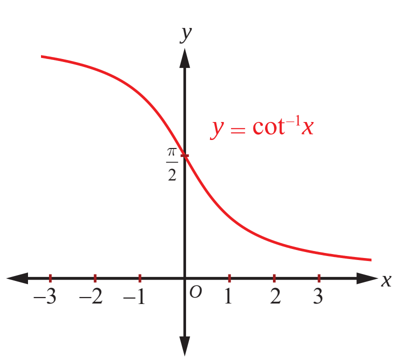


Fig. 4.30

4.9 Principal Value of Inverse Trigonometric Functions

Let us recall that the **principal value** of a inverse trigonometric function at a point x is the value of the inverse function at the point x, which lies in the range of principal branch. For instance, the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$, since $\frac{\pi}{6} \in [0, \pi]$. When there are two values, one is positive and the other is negative such that they are numerically equal, then the principal value of the inverse trigonometric function is the positive one. Now, we list out the **principal domain** and **range** of **trigonometric functions** and the **domain** and **range** of **inverse trigonometric functions**.

Function	Principal Domain	Range	Inverse Function	Domain	Range of Principal value branch
sine	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1,1]$	$\sin^{-1}$	$[-1,1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
cosine	$[0, \pi]$	$[-1,1]$	$\cos^{-1}$	$[-1,1]$	$[0, \pi]$
tangent	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\mathbb{R}$	$\tan^{-1}$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
cosecant	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$	$\mathbb{R} \setminus (-1, 1)$	cosec^{-1}	$\mathbb{R} \setminus (-1,1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$
secant	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$	$\mathbb{R} \setminus (-1,1)$	$\sec^{-1}$	$\mathbb{R} \setminus (-1,1)$	$[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$
cotangent	$(0, \pi)$	$\mathbb{R}$	$\cot^{-1}$	$\mathbb{R}$	$(0, \pi)$

Example 4.12

Find the principal value of

(i) $\text{cosec}^{-1}(-1)$ (ii) $\sec^{-1}(-2)$.

Solution

(i) Let $\text{cosec}^{-1}(-1) = y$. Then, $\text{cosec}\, y = -1$

Since the **range of principal value branch** of $y = \text{cosec}^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ and

$\text{cosec}\left(-\frac{\pi}{2}\right) = -1$, we have $y = -\frac{\pi}{2}$. Note that $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$.

Thus, the principal value of $\text{cosec}^{-1}(-1)$ is $-\frac{\pi}{2}$.

(ii) Let $y = \sec^{-1}(-2)$. Then, $\sec y = -2$.

By definition, the **range of the principal value branch** of $y = \sec^{-1}x$ is $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$.

Let us find y in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ such that $\sec y = -2$.

But, $\sec y = -2 \Rightarrow \cos y = -\frac{1}{2}$.

Now, $\cos y = -\frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$. Therefore, $y = \frac{2\pi}{3}$.

Since $\frac{2\pi}{3} \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$, the principal value of $\sec^{-1}(-2)$ is $\frac{2\pi}{3}$. ■

Example 4.13

Find the value of $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$.

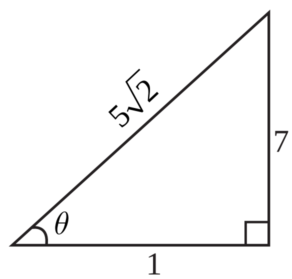
Solution

Let $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = \theta$. Then, $\sec\theta = -\frac{2}{\sqrt{3}}$ where $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$. Thus, $\cos\theta = -\frac{\sqrt{3}}{2}$.

Now, $\cos\frac{5\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. Hence, $\sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$. ■

Example 4.14

If $\cot^{-1}\left(\frac{1}{7}\right) = \theta$, find the value of $\cos\theta$.



Solution

By definition, $\cot^{-1} x \in (0, \pi)$.

Therefore, $\cot^{-1}\left(\frac{1}{7}\right) = \theta$ implies $\theta \in (0, \pi)$.

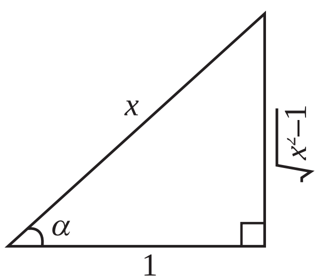
But $\cot^{-1}\left(\frac{1}{7}\right) = \theta$ implies $\cot\theta = \frac{1}{7}$ and hence $\tan\theta = 7$ and θ is acute. ■

Using $\tan\theta = \frac{7}{1}$, we construct a right triangle as shown . Then, we have, $\cos\theta = \frac{1}{5\sqrt{2}}$.

Example 4.15

Show that $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x$, $|x| > 1$.

Solution



Let $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \alpha$. Then, $\cot\alpha = \frac{1}{\sqrt{x^2-1}}$ and α is acute.

We construct a right triangle with the given data.

From the triangle, $\sec\alpha = \frac{x}{1} = x$. Thus, $\alpha = \sec^{-1} x$.

Hence, $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \sec^{-1} x$, $|x| > 1$. ■

EXERCISE 4.4

1. Find the principal value of

(i) $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (ii) $\cot^{-1}\left(\sqrt{3}\right)$ (iii) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

2. Find the value of

(i) $\tan^{-1}\left(\sqrt{3}\right)-\sec^{-1}(-2)$ (ii) $\sin^{-1}(-1)+\cos^{-1}\left(\frac{1}{2}\right)+\cot^{-1}(2)$

(iii) $\cot^{-1}(1)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-\sec^{-1}\left(-\sqrt{2}\right)$

4.10 Properties of Inverse Trigonometric Functions

In this section, we investigate some properties of inverse trigonometric functions. The properties to be discussed are valid within the principal value branches of the corresponding inverse trigonometric functions and where they are defined.

Property-I

(i) $\sin^{-1}(\sin\theta)=\theta$, if $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. (ii) $\cos^{-1}(\cos\theta)=\theta$, if $\theta\in[0,\pi]$.

(iii) $\tan^{-1}(\tan\theta)=\theta$, if $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. (iv) $\operatorname{cosec}^{1}(\operatorname{cosec}\theta)=\theta$, if $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$

(v) $\sec^{-1}(\sec\theta)=\theta$, if $\theta\in[0,\pi]\setminus\left\{\frac{\pi}{2}\right\}$. (vi) $\cot^{-1}(\cot\theta)=\theta$, if $\theta\in(0,\pi)$.

Proof

All the above results follow from the definition of the respective inverse functions.

For instance, (i) let $\sin\theta=x$; $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Now, $\sin\theta=x$ gives $\theta=\sin^{-1}x$, by definition of inverse sine function.

Thus, $\sin^{-1}(\sin\theta)=\theta$.

Property-II

(i) $\sin\left(\sin^{-1}x\right)=x$, if $x\in[-1,1]$. (ii) $\cos\left(\cos^{-1}x\right)=x$, if $x\in[-1,1]$

(iii) $\tan\left(\tan^{-1}x\right)=x$, if $x\in\mathbb{R}$ (iv) $\operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right)=x$, if $x\in\mathbb{R}\setminus(-1,1)$

(v) $\sec\left(\sec^{-1}x\right)=x$, if $x\in\mathbb{R}\setminus(-1,1)$ (vi) $\cot\left(\cot^{-1}x\right)=x$, if $x\in\mathbb{R}$

Proof

(i) For $x\in[-1,1]$, $\sin^{-1}x$ is well defined.

Let $\sin^{-1}x=\theta$. Then, by definition $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\theta=x$

Thus, $\sin\theta=x$ implies $\sin\left(\sin^{-1}x\right)=x$.

Similarly, other results are proved.

Note

(i) For any trigonometric function $y=f(x)$, we have $f\left(f^{-1}(x)\right)=x$ for all x in the range of f. This follows from the definition of $f^{-1}(x)$. When we have, $f\left(g^{-1}(x)\right)$, where $g^{-1}(x)=\sin^{-1}x$ or $\cos^{-1}x$, it will usually be necessary to draw a triangle defined by the inverse trigonometric function to solve the problem. For instance, to find $\cot\left(\sin^{-1}x\right)$, we have to draw a triangle using $\sin^{-1}x$. However, we have to be a little more careful with expression of the form $f^{-1}\left(f(x)\right)$.

(ii) Evaluation of $f^{-1}[f(x)]$, where f is any one of the six trigonometric functions.

(a) If x is in the restricted domain (principal domain) of f, then $f^{-1}[f(x)] = x$.

(b) If x is not in the restricted domain of f, then find x_1 within the restricted domain of f such that $f(x) = f(x_1)$. Now, $f^{-1}[f(x)] = x_1$. For instance,

$$\sin^{-1}(\sin x) = \begin{cases} x & \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ x_1 & \text{if } x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \text{where } \sin x = \sin x_1 \text{ and } x_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{cases}$$

Property-III (Reciprocal inverse identities)

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, if $x \in \mathbb{R} \setminus (-1, 1)$. (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, if $x \in \mathbb{R} \setminus (-1, 1)$.

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & \text{if } x > 0 \\ -\pi + \cot^{-1} x & \text{if } x < 0. \end{cases}$

Proof

(i) If $x \in \mathbb{R} \setminus (-1, 1)$, then $\frac{1}{x} \in [-1, 1]$ and $x \neq 0$. Thus, $\sin^{-1}\left(\frac{1}{x}\right)$ is well defined.

Let $\sin^{-1}\left(\frac{1}{x}\right) = \theta$. Then, by definition $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ and $\sin\theta = \frac{1}{x}$.

Thus, $\operatorname{cosec}\theta = x$, which in turn gives $\theta = \operatorname{cosec}^{-1} x$.

Now, $\sin^{-1}\left(\frac{1}{x}\right) = \theta = \operatorname{cosec}^{-1} x$. Thus, $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, $x \in \mathbb{R} \setminus (-1, 1)$.

Similarly, other results are proved.

Property-IV (Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, if $x \in [-1, 1]$.

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, if $x \in \mathbb{R}$.

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$.

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, if $x \in [-1, 1]$.

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$.

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, if $x \in \mathbb{R}$.

Proof

(i) If $x \in [-1, 1]$, then $-x \in [-1, 1]$. Thus, $\sin^{-1}(-x)$ is well defined

Let $\sin^{-1}(-x) = \theta$. Then $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = -x$.

Now, $\sin\theta = -x$ gives $x = -\sin\theta = \sin(-\theta)$

From $x=\sin(-\theta)$, we must have $\sin^{-1}x=-\theta$, which in turn gives $\theta=-\sin^{-1}x$.

Hence, $\sin^{-1}(-x)=-\sin^{-1}x$.

(iv) If $x\in[-1,1]$, then $-x\in[-1,1]$. Thus, $\cos^{-1}(-x)$ is well defined

Let $\cos^{-1}(-x)=\theta$. Then $\theta\in[0,\pi]$ and $\cos\theta=-x$.

Now, $\cos\theta=-x$ implies $x=-\cos\theta=\cos(\pi-\theta)$.

Thus, $\pi-\theta=\cos^{-1}x$, which gives $\theta=\pi-\cos^{-1}x$.

Hence, $\cos^{-1}(-x)=\pi-\cos^{-1}x$.

Similarly, other results are proved.

Note

(i) The inverse function of an one-to-one and odd function is also an odd function. For instance, $y=\sin^{-1}x$ is an odd function, since sine function is both one-to-one and odd in the restricted domain $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

(ii) Is the inverse function of an even function also even? It turns out that the question does not make sense, because an even function cannot be one-to-one if it is defined anywhere other than 0. Observe that $\cos^{-1}x$ and $\sec^{-1}x$ are not even functions.

Property-V (Cofunction inverse identities)

(i) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2},\ x\in[-1,1]$. (ii) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\ x\in\mathbb{R}$.

(iii) $\operatorname{cosec}^{-1}x+\sec^{-1}x=\frac{\pi}{2},\ x\in\mathbb{R}\setminus(-1,1)$ or $|x|\geq 1$.

Proof

(i) Here, $x\in[-1,1]$. Let $\sin^{-1}x=\theta$. Then $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $\sin\theta=x$.

Note that $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}\Leftrightarrow 0\leq\frac{\pi}{2}-\theta\leq\pi$.

So, $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=x$, which gives $\cos^{-1}x=\frac{\pi}{2}-\theta=\frac{\pi}{2}-\sin^{-1}x$.

Hence, $\cos^{-1}x+\sin^{-1}x=\frac{\pi}{2},\ |x|\leq 1$.

(ii) Let $\cot^{-1}x=\theta$. Then, $\cot\theta=x$, $0<\theta<\pi$ and $x\in\mathbb{R}$.

Now, $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta=x.$ ... (1)

Thus, for $x\in\mathbb{R}$, $\tan(\tan^{-1}x)=x$ and (1) gives $\tan(\tan^{-1}x)=\tan\left(\frac{\pi}{2}-\theta\right)$.

Hence, $\tan(\tan^{-1}x)=\tan\left(\frac{\pi}{2}-\cot^{-1}x\right)$ ... (2)

Note that $0<\cot^{-1}x<\pi$ gives $-\frac{\pi}{2}<\frac{\pi}{2}-\cot^{-1}x<\frac{\pi}{2}$.

Thus, (2) gives $\tan^{-1}x=\frac{\pi}{2}-\cot^{-1}x$. So, $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2},\ x\in\mathbb{R}$.

Similarly, (iii) can be proved.

Property-VI

(i) $\sin^{-1}x+\sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$, where either $x^2+y^2\le 1$ or $xy<0$.

(ii) $\sin^{-1}x-\sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$, where either $x^2+y^2\le 1$ or $xy>0$.

(iii) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x+y\ge 0$.

(iv) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}\left[xy+\sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x\le y$.

(v) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, if $xy<1$.

(vi) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$ if $xy>-1$.

Proof

(i) Let $A=\sin^{-1}x$. Then, $x=\sin A$; $A\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$; $|x|\le 1$ and $\cos A$ is positive

Let $B=\sin^{-1}y$. Then, $y=\sin B$; $B\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$; $|y|\le 1$ and $\cos B$ is positive

Now, $\cos A=+\sqrt{1-\sin^2 A}=\sqrt{1-x^2}$ and $\cos B=+\sqrt{1-\sin^2 B}=\sqrt{1-y^2}$

Thus, $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$= x\sqrt{1-y^2}+y\sqrt{1-x^2}$, where $|x|\le 1$; $|y|\le 1$ and hence, $x^2+y^2\le 1$

Therefore, $A+B=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$

Thus, $\sin^{-1}x+\sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)$, where either $x^2+y^2\le 1$ or $xy<0$.

Similarly, other results are proved.

Property-VII

(i) $2\tan^{-1}x=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right), |x|<1$

(ii) $2\tan^{-1}x=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, $x\ge 0$

(iii) $2\tan^{-1}x=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right), |x|\le 1$.

Proof

(i) By taking $y=x$ in Property-VI (v), we get the desired result

$$2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), |x|<1.$$

(ii) Let $\theta=2\tan^{-1}x$. Then, $\tan\dfrac{\theta}{2}=x$.

The identity $\cos\theta=\dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\dfrac{1-x^2}{1+x^2}$ gives $\theta=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

Hence, $2\tan^{-1} x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right),\ x \geq 0$.

Similarly, other result is proved.

Property-VIII

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$ if $|x| \leq \dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}} \leq x \leq \dfrac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x$ if $\dfrac{1}{\sqrt{2}} \leq x \leq 1$.

Proof

(i) Let $x = \sin\theta$.

Now, $2x\sqrt{1-x^2} = 2\sin\theta\cos\theta = \sin 2\theta$

Thus, $2\theta = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$. Hence, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1} x$.

(ii) Let $x = \cos\theta$.

Now, $2x\sqrt{1-x^2} = 2\cos\theta\sin\theta = \sin 2\theta$, which gives

$2\theta = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$. Hence, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1} x$.

Property-IX

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$. (ii) $\sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$.

(iii) $\sin^{-1} x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$. (iv) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$, if $0 \leq x \leq 1$.

(v) $\cos^{-1} x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1 \leq x < 0$. (vi) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$, if $x > 0$.

Proof

(i) Let $\sin^{-1} x = \theta$. Then, $\sin\theta = x$. Since $0 \leq x \leq 1$, we get $0 \leq \theta \leq \dfrac{\pi}{2}$.

$\cos\theta = \sqrt{1-x^2}$ or $\cos^{-1}\sqrt{1-x^2} = \theta = \sin^{-1} x$.

Thus, $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$, $0 \leq x \leq 1$.

(ii) Suppose that $-1 \leq x \leq 0$ and $\sin^{-1} x = \theta$. Then $-\dfrac{\pi}{2} \leq \theta < 0$.

So, $\sin\theta = x$ and $\cos(-\theta) = \sqrt{1-x^2}$ (since $\cos\theta > 0$)

Thus, $\cos^{-1}\sqrt{1-x^2} = -\theta = -\sin^{-1} x$. Hence, $\sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$.

Similarly, other results are proved.

Property-X

(i) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$. (ii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\dfrac{1}{2}, 1\right]$.

Proof

(i) Let $x=\sin\theta$. Thus, $\theta=\sin^{-1}x$.

Now, $3x-4x^3=3\sin\theta-4\sin^3\theta=\sin 3\theta$.

Thus, $\sin^{-1}(3x-4x^3)=3\theta=3\sin^{-1}x$.

The other result is proved in a similar way.

Remark

(i) $\sin^{-1}(\cos x)=\begin{cases}\dfrac{\pi}{2}-x, & \text{if } x\in[0,\pi]\\[2mm] \dfrac{\pi}{2}-y, & \text{if } x\notin[0,\pi] \text{ and } \cos x=\cos y, y\in[0,\pi]\end{cases}$

(ii) $\cos^{-1}(\sin x)=\begin{cases}\dfrac{\pi}{2}-x, & \text{if } x\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\\[2mm] \dfrac{\pi}{2}-y, & \text{if } x\notin\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right] \text{ and } \sin x=\sin y,\ y\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\end{cases}$

Example 4.16

Prove that $\dfrac{\pi}{2}\le\sin^{-1}x+2\cos^{-1}x\le\dfrac{3\pi}{2}$.

Solution

$$\sin^{-1}x+2\cos^{-1}x=\sin^{-1}x+\cos^{-1}x+\cos^{-1}x=\frac{\pi}{2}+\cos^{-1}x$$

We know that $0\le\cos^{-1}x\le\pi$. Thus, $\dfrac{\pi}{2}+0\le\cos^{-1}x+\dfrac{\pi}{2}\le\pi+\dfrac{\pi}{2}$.

Thus, $\dfrac{\pi}{2}\le\sin^{-1}x+2\cos^{-1}x\le\dfrac{3\pi}{2}$.

Example 4.17

Simplify (i) $\cos^{-1}\left(\cos\left(\dfrac{13\pi}{3}\right)\right)$ (ii) $\tan^{-1}\left(\tan\left(\dfrac{3\pi}{4}\right)\right)$

(iii) $\sec^{-1}\left(\sec\left(\dfrac{5\pi}{3}\right)\right)$ (iv) $\sin^{-1}[\sin 10]$

Solution

(i) $\cos^{-1}\left(\cos\left(\dfrac{13\pi}{3}\right)\right)$. The range of principal values of $\cos^{-1}x$ is $[0,\pi]$.

Since $\dfrac{13\pi}{3}\notin[0,\pi]$, we write $\dfrac{13\pi}{3}$ as $\dfrac{13\pi}{3}=4\pi+\dfrac{\pi}{3}$, where $\dfrac{\pi}{3}\in[0,\pi]$.

Now, $\cos\left(\dfrac{13\pi}{3}\right)=\cos\left(4\pi+\dfrac{\pi}{3}\right)=\cos\dfrac{\pi}{3}$.

Thus, $\cos^{-1}\left(\cos\left(\dfrac{13\pi}{3}\right)\right)=\cos^{-1}\left(\cos\left(\dfrac{\pi}{3}\right)\right)=\dfrac{\pi}{3}$, since $\dfrac{\pi}{3}\in[0,\pi]$.

(ii) $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$.

Observe that $\frac{3\pi}{4}$ is not in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the principal range of $\tan^{-1} x$.

So, we write $\frac{3\pi}{4} = \pi - \frac{\pi}{4}$.

Now, $\tan\left(\frac{3\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right) = -\tan\frac{\pi}{4} = \tan\left(-\frac{\pi}{4}\right)$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hence, $\tan^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{4}\right)\right) = -\frac{\pi}{4}$, since $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(iii) $\sec^{-1}\left(\sec\left(\frac{5\pi}{3}\right)\right)$.

Note that $\frac{5\pi}{3}$ is not in $[0, \pi]\setminus\left\{\frac{\pi}{2}\right\}$, the principal range of $\sec^{-1} x$.

We write $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$. Now, $\sec\left(\frac{5\pi}{3}\right) = \sec\left(2\pi - \frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right)$ and $\frac{\pi}{3} \in [0, \pi]\setminus\left\{\frac{\pi}{2}\right\}$.

Hence, $\sec^{-1}\left(\sec\left(\frac{5\pi}{3}\right)\right) = \sec^{-1}\left(\sec\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

(iv) $\sin^{-1}[\sin 10]$

We know that $\sin^{-1}(\sin\theta) = \theta$ if $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Considering the approximation $\frac{\pi}{2} \simeq \frac{11}{7}$,

we conclude that $10 \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $(10-3\pi) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Now, $\sin 10 = \sin(3\pi + (10-3\pi)) = \sin(\pi + (10-3\pi)) = -\sin(10-3\pi) = \sin(3\pi - 10)$.

Hence, $\sin^{-1}[\sin 10] = \sin^{-1}[\sin(3\pi - 10)] = 3\pi - 10$, since $(3\pi - 10) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. ■

Example 4.18

Find the value of (i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ (ii) $\cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right]$

(iii) $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$.

Solution

(i) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right] = \sin\left(\frac{\pi}{2}\right) = 1$.

(ii) Consider $\cos\left[\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)\right]$. Let $\cos^{-1}\left(\frac{1}{8}\right) = \theta$. Then, $\cos\theta = \frac{1}{8}$ and $\theta \in [0, \pi]$.

Now, $\cos\theta = \dfrac{1}{8}$ implies $2\cos^2\dfrac{\theta}{2} - 1 = \dfrac{1}{8}$. Thus, $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{3}{4}$, since $\cos\left(\dfrac{\theta}{2}\right)$ is positive.

Thus, $\cos\left[\dfrac{1}{2}\cos^{-1}\left(\dfrac{1}{8}\right)\right] = \cos\left(\dfrac{\theta}{2}\right) = \dfrac{3}{4}$.

(iii) $\tan\left[\dfrac{1}{2}\sin^{-1}\left(\dfrac{2a}{1+a^2}\right) + \dfrac{1}{2}\cos^{-1}\left(\dfrac{1-a^2}{1+a^2}\right)\right]$

Let $a = \tan\theta$.

Now,

$$\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right] = \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\right]$$

$$= \tan\left[\frac{1}{2}\sin^{-1}(\sin 2\theta) + \frac{1}{2}\cos^{-1}(\cos 2\theta)\right] = \tan[2\theta] = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2a}{1-a^2}.$$

Example 4.19

Prove that $\tan(\sin^{-1}x) = \dfrac{x}{\sqrt{1-x^2}}$ for $|x| < 1$.

Solution

Let $\sin^{-1}x = \theta$. Then, $x = \sin\theta$ and $-1 \le x \le 1$

Now, $\tan(\sin^{-1}x) = \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \dfrac{x}{\sqrt{1-x^2}}$, $|x| < 1$.

Example 4.20

Evaluate $\sin\left[\sin^{-1}\left(\dfrac{3}{5}\right) + \sec^{-1}\left(\dfrac{5}{4}\right)\right]$

Solution

Let $\sec^{-1}\dfrac{5}{4} = \theta$. Then, $\sec\theta = \dfrac{5}{4}$ and hence, $\cos\theta = \dfrac{4}{5}$.

Also, $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\dfrac{4}{5}\right)^2} = \dfrac{3}{5}$, which gives $\theta = \sin^{-1}\left(\dfrac{3}{5}\right)$.

Thus, $\sec^{-1}\left(\dfrac{5}{4}\right) = \sin^{-1}\left(\dfrac{3}{5}\right)$ and $\sin^{-1}\dfrac{3}{5} + \sec^{-1}\left(\dfrac{5}{4}\right) = 2\sin^{-1}\left(\dfrac{3}{5}\right)$.

We know that $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$, if $|x| \le \dfrac{1}{\sqrt{2}}$.

Since $\dfrac{3}{5} < \dfrac{1}{\sqrt{2}}$, we have $2\sin^{-1}\left(\dfrac{3}{5}\right) = \sin^{-1}\left(2\times\dfrac{3}{5}\sqrt{1-\left(\dfrac{3}{5}\right)^2}\right) = \sin^{-1}\left(\dfrac{24}{25}\right)$.

Hence, $\sin\left[\sin^{-1}\left(\dfrac{3}{5}\right) + \sec^{-1}\left(\dfrac{5}{4}\right)\right] = \sin\left(\sin^{-1}\left(\dfrac{24}{25}\right)\right) = \dfrac{24}{25}$, since $\dfrac{24}{25} \in [-1, 1]$.

Example 4.21

Prove that (i) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$ (ii) $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution

(i) We know that $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}$, $xy<1$.

$$\text{Thus,}\quad \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}=\tan^{-1}(1)=\frac{\pi}{4}.$$

(ii) We know that $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$, $-1<x<1$

$$\text{So,}\quad 2\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}=\tan^{-1}\left(\frac{4}{3}\right).$$

$$\text{Hence,}\quad 2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}\right)=\tan^{-1}\left(\frac{31}{17}\right).$$

■

Example 4.22

If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ and $0<x,y,z<1$, show that

$x^2+y^2+z^2+2xyz=1$

Solution

Let $\cos^{-1}x=\alpha$ and $\cos^{-1}y=\beta$. Then, $x=\cos\alpha$ and $y=\cos\beta$.

$\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$ gives $\alpha+\beta=\pi-\cos^{-1}z$. ... (1)

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=xy-\sqrt{1-x^2}\sqrt{1-y^2}$.

From (1), we get $\cos\left(\pi-\cos^{-1}z\right)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$

$-\cos\left(\cos^{-1}z\right)=xy-\sqrt{1-x^2}\sqrt{1-y^2}$.

So, $-z=xy-\sqrt{1-x^2}\sqrt{1-y^2}$, which gives $-xy-z=-\sqrt{1-x^2}\sqrt{1-y^2}$.

Squaring on both sides and simplifying, we get $x^2+y^2+z^2+2xyz=1$. ■

Example 4.23

If $a_1, a_2, a_3, \ldots a_n$ is an arithmetic progression with common difference d,

prove that $\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}$.

Solution

Now, $\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)=\tan^{-1}\left(\frac{a_2-a_1}{1+a_la_2}\right)=\tan^{-1}a_2-\tan^{-1}a_1$.

Similarly, $\tan^{-1}\left(\dfrac{d}{1+a_2a_3}\right) = \tan^{-1}\left(\dfrac{a_3-a_2}{1+a_2a_3}\right) = \tan^{-1}a_3 - \tan^{-1}a_2$.

Continuing inductively, we get

$$\tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right) = \tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_{n-1}a_n}\right) = \tan^{-1}a_n - \tan^{-1}a_{n-1}.$$

Adding vertically, we get

$$\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right) = \tan^{-1}a_n - \tan^{-1}a_1.$$

$$\tan\left[\tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + \tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \ldots + \tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right] = \tan[\tan^{-1}a_n - \tan^{-1}a_1]$$

$$= \tan\left[\tan^{-1}\left(\frac{a_n-a_1}{1+a_la_n}\right)\right] = \frac{a_n-a_1}{1+a_la_n}.$$

Example 4.24 Solve $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1}x$ for $x > 0$.

Solution

$\tan^{-1}\left(\dfrac{1-x}{1+x}\right) = \dfrac{1}{2}\tan^{-1}x$ gives $\tan^{-1}1 - \tan^{-1}x = \dfrac{1}{2}\tan^{-1}x$.

Therefore, $\dfrac{\pi}{4} = \dfrac{3}{2}\tan^{-1}x$, which in turn reduces to $\tan^{-1}x = \dfrac{\pi}{6}$

Thus, $x = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$.

Example 4.25 Solve $\sin^{-1}x > \cos^{-1}x$

Solution

Given that $\sin^{-1}x > \cos^{-1}x$. Note that $-1 \le x \le 1$.

Adding both sides by $\sin^{-1}x$, we get

$\sin^{-1}x + \sin^{-1}x > \cos^{-1}x + \sin^{-1}x$, which reduces to $2\sin^{-1}x > \dfrac{\pi}{2}$.

As sine function increases in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we have $x > \sin\dfrac{\pi}{4}$ or $x > \dfrac{1}{\sqrt{2}}$.

Thus, the solution set is the interval $\left(\dfrac{1}{\sqrt{2}}, 1\right]$.

Example 4.26

Show that $\cot(\sin^{-1}x) = \dfrac{\sqrt{1-x^2}}{x}, -1 \le x \le 1$ and $x \ne 0$

Solution

Let $\sin^{-1}x = \theta$. Then, $x = \sin\theta$ and $x \ne 0$, we get $\theta \in \left[\dfrac{-\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$.

Hence, $\cos\theta \ge 0$ and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$.

Thus, $\cot(\sin^{-1}x) = \cot\theta = \dfrac{\sqrt{1-x^2}}{x}$, $|x| \le 1$ and $x \ne 0$

Example 4.27: Solve $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$, if $6x^2<1$.

Solution

Now, $\tan^{-1}2x+\tan^{-1}3x = \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right)$, since $6x^2<1$.

So, $\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$, which implies $\frac{5x}{1-6x^2}=\tan\frac{\pi}{4}=1$.

Thus, $1-6x^2 = 5x$, which gives $6x^2+5x-1=0$

Hence, $x = \frac{1}{6},-1$. But $x=-1$ does not satisfy $6x^2<1$.

Observe that $x=-1$ makes the left side of the equation negative whereas the right side is a positive number. Thus, $x=-1$ is not a solution. Hence, $x=\frac{1}{6}$ is the only solution of the equation. ■

Example 4.28

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$.

Solution

Now, $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$.

Thus, $\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)} = 1$, which on simplification gives $2x^2-4=-3$

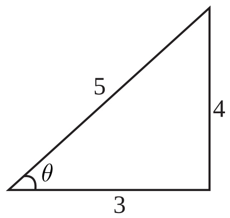
Thus, $x^2 = \frac{1}{2}$ gives $x=\pm\frac{1}{\sqrt{2}}$.

■

Example 4.29

Solve $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}$.

Solution



We know that $\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$.

Thus, $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \frac{1}{\sqrt{1+x^2}}$. ...(1)

Let $\cot^{-1}\left(\frac{3}{4}\right)=\theta$. Then $\cot\theta=\frac{3}{4}$ and so θ is acute.

From the diagram, we get

Hence, $\sin\left\{\cot^{-1}\left(\frac{3}{4}\right)\right\}=\sin\theta=\frac{4}{5}$ ... (2)

Using (1) and (2) in the given equation, we get $\frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$, which gives $\sqrt{1+x^2} = \frac{5}{4}$

Thus, $x=\pm\frac{3}{4}$. ■

EXERCISE 4.5

1. Find the value, if it exists. If not, give the reason for non-existence.

   (i) $\sin^{-1}(\cos\pi)$ (ii) $\tan^{-1}\left(\sin\left(-\frac{5\pi}{2}\right)\right)$ (iii) $\sin^{-1}[\sin 5]$.

2. Find the value of the expression in terms of x, with the help of a reference triangle.

   (i) $\sin\left(\cos^{-1}(1-x)\right)$ (iii) $\cos\left(\tan^{-1}(3x-1)\right)$ (iii) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$.

3. Find the value of

   (i) $\sin^{-1}\left(\cos\left(\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$ (ii) $\cot\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{4}{5}\right)$ (iii) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$.

4. Prove that

   (i) $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$ (ii) $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}=\sin^{-1}\frac{16}{65}$.

5. Prove that $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$.

6. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that $x+y+z=xyz$.

7. Prove that $\tan^{-1}x+\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{3x-x^3}{1-3x^2}$, $|x|<\frac{1}{\sqrt{3}}$.

8. Simplify: $\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{x-y}{x+y}$.

9. Solve:

   (i) $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}$ (ii) $2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}$, $a>0,\ b>0$.

   (iii) $2\tan^{-1}(\cos x)=\tan^{-1}(2\operatorname{cosec} x)$ (iv) $\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}$, $x>0$.

10. Find the number of solutions of the equation $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}(3x)$.

EXERCISE 4.6

Choose the correct or the most suitable answer from the given four alternatives.

1. The value of $\sin^{-1}(\cos x)$, $0\le x\le\pi$ is

   (1) $\pi-x$ (2) $x-\frac{\pi}{2}$ (3) $\frac{\pi}{2}-x$ (4) $x-\pi$

2. If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$; then $\cos^{-1}x+\cos^{-1}y$ is equal to

   (1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) π

3. $\sin^{-1}\frac{3}{5}-\cos^{-1}\frac{12}{13}+\sec^{-1}\frac{5}{3}-\text{cosec}^{-1}\frac{13}{12}$ is equal to

(1) 2π (2) π (3) 0 (4) $\tan^{-1}\frac{12}{65}$

4. If $\sin^{-1}x=2\sin^{-1}\alpha$ has a solution, then

(1) $|\alpha|\le\frac{1}{\sqrt{2}}$ (2) $|\alpha|\ge\frac{1}{\sqrt{2}}$ (3) $|\alpha|<\frac{1}{\sqrt{2}}$ (4) $|\alpha|>\frac{1}{\sqrt{2}}$

5. $\sin^{-1}(\cos x)=\frac{\pi}{2}-x$ is valid for

(1) $-\pi\le x\le 0$ (2) $0\le x\le\pi$ (3) $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$ (4) $-\frac{\pi}{4}\le x\le\frac{3\pi}{4}$

6. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, the value of $x^{2017}+y^{2018}+z^{2019}-\frac{9}{x^{101}+y^{101}+z^{101}}$ is

(1) 0 (2) 1 (3) 2 (4) 3

7. If $\cot^{-1}x=\frac{2\pi}{5}$ for some $x\in R$, the value of $\tan^{-1}x$ is

(1) $-\frac{\pi}{10}$ (2) $\frac{\pi}{5}$ (3) $\frac{\pi}{10}$ (4) $-\frac{\pi}{5}$

8. The domain of the function defined by $f(x)=\sin^{-1}\sqrt{x-1}$ is

(1) $[1,\ 2]$ (2) $[-1,\ 1]$ (3) $[0,1]$ (4) $[-1,\ 0]$

9 If $x=\frac{1}{5}$, the value of $\cos\left(\cos^{-1}x+2\sin^{-1}x\right)$ is

(1) $-\sqrt{\frac{24}{25}}$ (2) $\sqrt{\frac{24}{25}}$ (3) $\frac{1}{5}$ (4) $-\frac{1}{5}$

10. $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$ is equal to

(1) $\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$ (2) $\frac{1}{2}\sin^{-1}\left(\frac{3}{5}\right)$ (3) $\frac{1}{2}\tan^{-1}\left(\frac{3}{5}\right)$ (4) $\tan^{-1}\left(\frac{1}{2}\right)$

11. If the function $f(x)=\sin^{-1}\left(x^2-3\right)$, then x belongs to

(1) $[-1,1]$ (2) $\left[\sqrt{2},2\right]$

(3) $\left[-2,-\sqrt{2}\right]\cup\left[\sqrt{2},\ 2\right]$ (4) $\left[-2,-\sqrt{2}\right]$

12. If $\cot^{-1}2$ and $\cot^{-1}3$ are two angles of a triangle, then the third angle is

(1) $\frac{\pi}{4}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$

13. $\sin^{-1}\left(\tan\frac{\pi}{4}\right)-\sin^{-1}\left(\sqrt{\frac{3}{x}}\right)=\frac{\pi}{6}$. Then x is a root of the equation

(1) $x^2-x-6=0$ (2) $x^2-x-12=0$ (3) $x^2+x-12=0$ (4) $x^2+x-6=0$

14. $\sin^{-1}\left(2\cos^2 x-1\right)+\cos^{-1}\left(1-2\sin^2 x\right)=$

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

15. If $\cot^{-1}\left(\sqrt{\sin\alpha}\right)+\tan^{-1}\left(\sqrt{\sin\alpha}\right)=u$, then $\cos 2u$ is equal to

(1) $\tan^2\alpha$ (2) 0 (3) -1 (4) $\tan 2\alpha$

16. If $|x|\leq 1$, then $2\tan^{-1}x-\sin^{-1}\frac{2x}{1+x^2}$ is equal to

(1) $\tan^{-1}x$ (2) $\sin^{-1}x$ (3) 0 (4) π

17. The equation $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

(1) no solution (2) unique solution

(3) two solutions (4) infinite number of solutions

18. If $\sin^{-1}x+\cot^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{2}$, then x is equal to

(1) $\frac{1}{2}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{3}}{2}$

19. If $\sin^{-1}\frac{x}{5}+\operatorname{cosec}^{-1}\frac{5}{4}=\frac{\pi}{2}$, then the value of x is

(1) 4 (2) 5 (3) 2 (4) 3

20. $\sin(\tan^{-1}x), |x|<1$ is equal to

(1) $\frac{x}{\sqrt{1-x^2}}$ (2) $\frac{1}{\sqrt{1-x^2}}$ (3) $\frac{1}{\sqrt{1+x^2}}$ (4) $\frac{x}{\sqrt{1+x^2}}$

SUMMARY

Inverse Trigonometric Functions

Inverse sine function	Inverse cosine function	Inverse tangent function	Inverse cosecant function	Inverse secant function	Inverse cot function
Domain $[-1,1]$	Domain $[-1,1]$	Domain $\mathbb{R}$	Domain $(-\infty,-1]\cup[1,\infty)$	Domain $(-\infty,-1]\cup[1,\infty)$	Domain $\mathbb{R}$
Range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	Range $[0,\pi]$	Range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	Range $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$	Range $[0,\pi]-\left\{\frac{\pi}{2}\right\}$	Range $(0,\pi)$
not a periodic function	not a periodic function	not a periodic function	not a periodic function	not a periodic function	not a periodic function
odd function	neither even nor odd function	odd function	odd function	neither even nor odd function	neither even nor odd function
strictly increasing function	strictly decreasing function	strictly increasing function	strictly decreasing function with respect to its domain.	strictly decreasing function with respect to its domain.	strictly decreasing function
one to one function	one to one function	one to one function	one to one function	one to one function	one to one function

Properties of Inverse Trigonometric Functions

Property-I

(i) $\sin^{-1}(\sin\theta)=\theta$, if $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (ii) $\cos^{-1}(\cos\theta)=\theta$, if $\theta\in[0,\pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta$, if $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (iv) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta$, if $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\setminus\{0\}$

(v) $\sec^{-1}(\sec\theta)=\theta$, if $\theta\in[0,\pi]\setminus\left\{\frac{\pi}{2}\right\}$ (vi) $\cot^{-1}(\cot\theta)=\theta$, if $\theta\in(0,\pi)$

Property-II

(i) $\sin\left(\sin^{-1}x\right)=x$, if $x\in[-1,1]$ (ii) $\cos\left(\cos^{-1}x\right)=x$, if $x\in[-1,1]$

(iii) $\tan\left(\tan^{-1}x\right)=x$, if $x\in\mathbb{R}$ (iv) $\text{cosec}\left(\text{cosec}^{-1}x\right)=x$, if $x\in\mathbb{R}\setminus(-1,1)$

(v) $\sec\left(\sec^{-1}x\right)=x$, if $x\in\mathbb{R}\setminus(-1,1)$ (vi) $\cot\left(\cot^{-1}x\right)=x$, if $x\in\mathbb{R}$

Property-III (Reciprocal inverse identities)

(i) $\sin^{-1}\left(\frac{1}{x}\right)=\text{cosec}\,x$, if $x\in\mathbb{R}\setminus(-1,1)$.   (ii) $\cos^{-1}\left(\frac{1}{x}\right)=\sec x$, if $x\in\mathbb{R}\setminus(-1,1)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right)=\begin{cases}\cot^{-1}x & \text{if } x>0\\ -\pi+\cot^{-1}x & \text{if } x<0.\end{cases}$

Property-IV(Reflection identities)

(i) $\sin^{-1}(-x) = -\sin^{-1}x$,if $x \in [-1, 1]$.

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$,if $x \in \mathbb{R}$.

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$,if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$.

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1}x$,if $x \in [-1, 1]$.

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$,if $|x| \geq 1$ or $x \in \mathbb{R} \setminus (-1, 1)$.

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, if $x \in \mathbb{R}$.

Property-V (Cofunction inverse identities)

(i) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2},\ x \in [-1, 1].$ (ii) $\tan^{-1}x + \cot^{-1}x = \dfrac{\pi}{2},\ x \in \mathbb{R}.$

(iii) $\operatorname{cos}ec^{-1}x + \sec^{-1}x = \dfrac{\pi}{2},\ x \in \mathbb{R} \setminus (-1, 1)$ or $|x| \geq 1$.

Property-VI

(i) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$, where either $x^2 + y^2 \leq 1$ or $xy < 0$.

(ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, where either $x^2 + y^2 \leq 1$ or $xy > 0$.

(iii) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x + y \geq 0$.

(iv) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$, if $x\,''\,y$.

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$, if $xy < 1$.

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$, if $xy > -1$.

Property-VII

(i) $2\tan^{-1}x = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right),\ |x| < 1$ (ii) $2\tan^{-1}x = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right),\ x \geq 0$

(iii) $2\tan^{-1}x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right),\ |x| \leq 1$

Property-VIII

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$, if $|x| \leq \dfrac{1}{\sqrt{2}}$ or $-\dfrac{1}{\sqrt{2}} \leq x \leq \dfrac{1}{\sqrt{2}}$.

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$, if $\dfrac{1}{\sqrt{2}} \leq x \leq 1$.

Property-IX

(i) $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2}$, if $0 \le x \le 1$. (ii) $\sin^{-1} x = -\cos^{-1}\sqrt{1-x^2}$, if $-1 \le x < 0$.

(ii) $\sin^{-1} x = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$, if $-1 < x < 1$. (iv) $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$, if $0 \le x \le 1$.

(v) $\cos^{-1} x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1 \le x < 0$. (vi) $\tan^{-1} x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$, if $x > 0$.

Property-X

(i) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$. (ii) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $x \in \left[\dfrac{1}{2}, 1\right]$.

ICT CORNER

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NEVBT

Open the Browser, type the URL Link given below (or) Scan the QR code. GeoGebra work book named "12th Standard Mathematics" will open. In the left side of the work book there are many chapters related to your text book. Click on the chapter named "Inverse Trigonometric Functions". You can see several work sheets related to the chapter. Select the work sheet "Graph of Inverse Trigonometric Functions"

Chapter

5

Two Dimensional Analytical Geometry-II

"Divide each difficulty into as many parts as is feasible and necessary to resolve it"

René Descartes

René Descartes
1596 – 1650

5.1 Introduction

Analytical Geometry of two dimension is used to describe geometric objects such as **point, line, circle, parabola, ellipse,** and **hyperbola** using **Cartesian coordinate system**.Two thousand years ago (≈ 2– 1 BC (BCE)), the ancient Greeks studied **conic** curves, because studying them elicited ideas that were exciting, challenging, and interesting. They could not have imagined the applications of these curves in the later centuries.

Solving problems by the method of Analytical Geometry was systematically developed in the first half of the 17th century majorly, by Descartes and also by other great mathematicians like Fermat, Kepler, Newton, Euler, Leibniz, l'Hôpital, Clairaut, Cramer, and the Jacobis.

Analytic Geometry grew out of need for establishing **algebraic techniques** for solving **geometrical problems** and the development in this area has conquered industry, medicine, and scientific research.

The theory of Planetary motions developed by Johannes Kepler, the German mathematician cum physicist stating that all the planets in the solar system including the earth are moving in elliptical orbits with Sun at one of a foci, governed by inverse square law paved way to established work in Euclidean geometry. Euler applied the co-ordinate method in a systematic study of space curves and surfaces, which was further developed by Albert Einstein in his theory of relativity.

Applications in various fields encompassing **gears**, **vents** in dams, **wheels** and circular geometry leading to trigonometry as application based on properties of circles; arches, dish, **solar cookers**, **head-lights**, **suspension bridges,** and **search lights** as application based on properties of parabola; arches, **Lithotripsy** in the field of Medicine, **whispering galleries,** Ne-de-yag lasers and gears as application based on properties of ellipse; and telescopes, cooling towers, **spotting locations** of ships or aircrafts as application based on properties of hyperbola, to name a few.

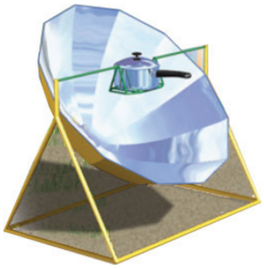

Fig. 5.1

Fig. 5.2

Fig. 5.3

Fig. 5.4

Fig. 5.5

A driver took the job of delivering a truck of books ordered on line. The truck is of $3m$ wide and $2.7m$ high, while driving he noticed a sign at the semielliptical entrance of a tunnel; Caution! Tunnel is of $3m$ high at the centre peak. Then he saw another sign; Caution! Tunnel is of $12m$ wide. Will his truck pass through the opening of tunnel's archway? We will be able to answer this question at the end of this chapter.

Fig. 5.6

Learning Objectives

Upon completion of this chapter, students will be able to

- write the equations of circle, parabola, ellipse, hyperbola in standard form,
- find the centre, vertices, foci etc. from the equation of different conics,
- derive the equations of tangent and normal to different conics,
- classify the conics and their degenerate forms,
- form equations of conics in parametric form, and their applications.
- apply conics in various real life situations.

5.2 Circle

The word circle is of Greek origin and reference to circles is found as early as 1700 BC (BCE). In Nature circles would have been observed, such as the Moon, Sun, and **ripples in water**. The circle is the basis for the wheel, which, with related inventions such as gears, makes much of modern machinery possible. In mathematics, the study of the circle has helped to inspire the development of geometry, **astronomy** and calculus. In Bohr-Sommerfeld theory of the atom, **electron orbit** is modelled as circle.

Definition 5.1

A circle is the locus of a point in a plane which moves such that its distance from a fixed point in the plane is always a constant.

The fixed point is called the **centre** and the constant distance is called **radius** of the circle.

5.2.1 Equation of a circle in standard form

(i) Equation of circle with centre (0, 0) and radius r

Let the centre be $C(0,0)$ and radius be r and $P(x,y)$ be the moving point.

Note that the point P having coordinates (x,y) is represented as $P(x,y)$.

$$\text{Then,} \quad CP = r \quad \text{and so } CP^2 = r^2$$

$$\text{Therfore} \quad (x-0)^2 + (y-0)^2 = r^2$$

$$\text{That is } x^2 + y^2 = r^2$$

This is the equation of the circle with centre $(0,0)$ and radius r.

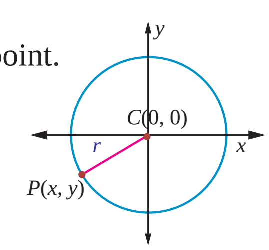

Fig.5.7

(ii) Equation of circle with centre (h, k) and radius r

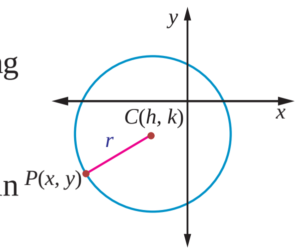

Fig.5.8

Let the centre be $C(h,k)$ and r be the radius and $P(x,y)$ be the moving point.

Then, $CP=r$ and so $CP^2=r^2$.

That is, $(x-h)^2+(y-k)^2=r^2$.This is the equation of the circle in Standard form, which is also known as centre-radius form.

Expanding the equation, we get

$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$.

Taking $2g=-2h, 2f=-2k, c=h^2+k^2-r^2$, the equation takes the form

$x^2+y^2+2gx+2fy+c=0$, called the general form of a circle.

The equation $x^2+y^2+2gx+2fy+c=0$ is a second degree equation in x and y possessing the following characteristics:

(i) It is a second degree equation in x and y,

(ii) coefficient of x^2 = coefficient of $y^2 \neq 0$,

(iii) coefficient of $xy=0$.

Conversely, we prove that an equation possessing these three characteristics, always represents a circle. Let

$$ax^2+ay^2+2g'x+2f'y+c=0 \qquad \ldots (1)$$

be a second degree equation in x and y having characteristics (i), (ii) and (iii), $a\neq 0$. Dividing (1) by a, gives

$$x^2+y^2+\frac{2g'}{a}x+\frac{2f'}{a}y+\frac{c'}{a}=0. \qquad \ldots (2)$$

Taking $\frac{g'}{a}=g, \frac{f'}{a}=f$ and $\frac{c'}{a}=c$, equation (2) becomes $x^2+y^2+2gx+2fy+c=0$.

Adding and subtracting g^2 and f^2, we get $x^2+2gx+g^2+y^2+2fy+f^2-g^2-f^2+c=0$

$$\Rightarrow (x+g)^2+(y+f)^2=g^2+f^2-c$$

$$\Rightarrow (x-(-g))^2+(y-(-f))^2=\left(\sqrt{g^2+f^2-c}\right)^2$$

This is in the standard form of a circle with centre C $(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$. Hence equation (1) represents a circle with centre $(-g,-f)=\left(\frac{-g'}{a},\frac{-f'}{a}\right)$ and radius

$$=\sqrt{g^2+f^2-c}=\frac{1}{a}\sqrt{g'^2+f'^2-c'a}.$$

Note

The equation of the circle $x^2+y^2+2gx+2fy+c=0$ with centre $(-g,-f)$ and radius $\sqrt{g^2+f^2-c}$ represents.

(i) a real circle if $g^2+f^2-c>0$;

(ii) a point circle if $g^2+f^2-c=0$;

(iii) an imaginary circle if $g^2+f^2-c<0$ with no locus.

Example 5.1

Find the general equation of a circle with centre $(-3,-4)$ and radius 3 units.

Solution

Equation of the circle in standard form is $(x-h)^2+(y-k)^2=r^2$

$$\Rightarrow \quad (x-(-3))^2+(y-(-4))^2 = 3^2$$
$$\Rightarrow \quad (x+3)^2+(y+4)^2 = 3^2$$
$$\Rightarrow \quad x^2+y^2+6x+8y+16 = 0.$$

Theorem 5.1

The circle passing through the points of intersection (real or imaginary) of the line $lx+my+n=0$ and the circle $x^2+y^2+2gx+2fy+c=0$ is the circle of the form
$x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0,\ \lambda\in\mathbb{R}^1$.

Proof

Let the circle be S : $x^2+y^2+2gx+2fy+c = 0$, ... (1)

and the line be L : $lx+my+n = 0$. ... (2)

Consider $S+\lambda L=0$. That is $x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0$ ... (3)

Grouping the terms of x, y and constants, we get

$x^2+y^2+x(2g+\lambda l)+y(2f+\lambda m)+c+\lambda n=0$ which is a second degree equation in x and y with coefficients of x^2 and y^2 equal and there is no xy term.

If (α,β) is a point of intersection of S and L satisfying equation (1) and (2), then it satisfies equation (3).

Hence $S+\lambda L=0$ represents the required circle.

Example 5.2

Find the equation of the circle described on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter.

Solution

Equation of the circle passing through the points of intersection of the chord and circle by Theorem 5.1 is $x^2+y^2-16+\lambda(3x+y+5)=0$.

The chord $3x+y+5=0$ is a diameter of this circle if the centre $\left(\frac{-3\lambda}{2},\frac{-\lambda}{2}\right)$ lies on the chord.

$$\text{So, we have } 3\left(\frac{-3\lambda}{2}\right)-\frac{\lambda}{2}+5 = 0,$$
$$\Rightarrow \quad \frac{-9\lambda}{2}-\frac{\lambda}{2}+5 = 0,$$
$$\Rightarrow \quad -5\lambda+5 = 0,$$
$$\Rightarrow \quad \lambda = 1.$$

Therefore, the equation of the required circle is $x^2+y^2+3x+y-11=0$.

Example 5.3

Determine whether $x+y-1=0$ is the equation of a diameter of the circle

$x^2+y^2-6x+4y+c=0$ for all possible values of c.

Solution

Centre of the circle is $(3,-2)$ which lies on $x+y-1=0$. So the line $x+y-1=0$ passes through the centre and therefore the line $x+y-1=0$ is a diameter of the circle for all possible values of c.

Theorem 5.2

The equation of a circle with (x_1, y_1) and (x_2, y_2) as extremities of one of the diameters of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

Proof

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the two extremities of the diameter AB, and $P(x, y)$ be any point on the circle. Then $\angle APB = \frac{\pi}{2}$ (angle in a semi-circle).

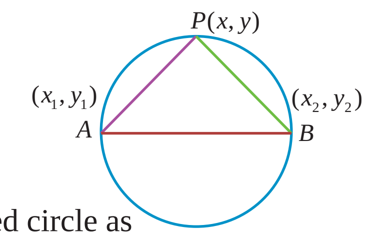

Fig.5.9

Therefore, the product of slopes of AP and PB is equal to -1.

That is, $\left(\frac{(y-y_1)}{(x-x_1)}\right)\left(\frac{(y-y_2)}{(x-x_2)}\right) = -1$ yielding the equation of the required circle as

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

Example 5.4

Find the general equation of the circle whose diameter is the line segment joining the points $(-4,-2)$ and $(1,1)$.

Solution

Equation of the circle with end points of the diameter as (x_1, y_1) and (x_2, y_2) given in theorem 5.2 is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$

$$\Rightarrow \quad (x+4)(x-1)+(y+2)(y-1) = 0$$

$\Rightarrow \quad x^2+y^2+3x+y-6=0$ which is the required equation of the circle.

Theorem 5.3

The position of a point $P(x_1, y_1)$ with respect to a given circle $x^2+y^2+2gx+2fy+c=0$ in the plane containing the circle is outside or on or inside the circle according as

$$x_1^2+y_1^2+2gx_1+2fy_1+c \text{ is } \begin{cases} >0 & \text{or,} \\ =0 & \text{or,} \\ <0. \end{cases}$$

Proof

Equation of the circle is $x^2+y^2+2gx+2fy+c=0$ with centre C $(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$.

Let $P(x_1, y_1)$ be a point in the plane. Join CP and let it meet the circle at Q. Then the point P is outside, on or within the circle according as

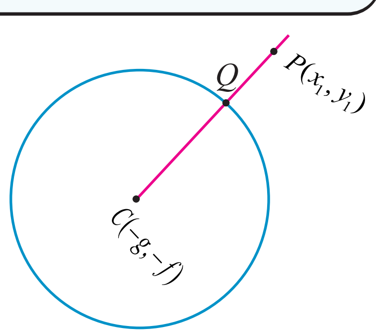

Fig.5.10

$$|CP| \text{ is } \begin{cases} >|CQ| & \text{or,} \\ =|CQ| & \text{or,} \\ <|CQ|. \end{cases}$$

$$\Rightarrow \qquad CP^2 \text{ is } \begin{cases} > r^2 & \text{or,} \\ = r^2 & \text{or} \quad \{CQ = r\}, \\ < r^2. \end{cases}$$

$$\Rightarrow \quad (x_1+g)^2 + (y_1+f)^2 \text{ is } \begin{cases} > g^2+f^2-c & \text{or,} \\ = g^2+f^2-c & \text{or,} \\ < g^2+f^2-c. \end{cases}$$

$$\Rightarrow \quad x_1^{\,2} + y_1^{\,2} + 2gx_1 + 2fy_1 + c \text{ is } \begin{cases} > 0 & \text{or,} \\ = 0 & \text{or,} \\ < 0. \end{cases}$$

Example 5.5

Examine the position of the point $(2,3)$ with respect to the circle $x^2+y^2-6x-8y+12=0$.

Solution

Taking (x_1, y_1) as (2, 3), we get

$$x_1^{\,2}+y_1^{\,2}+2gx_1+2fy_1+c = 2^2+3^2-6\times 2-8\times 3+12,$$
$$= 4+9-12-24+12 \;=-11<0.$$

Therefore, the point $(2,3)$ lies inside the circle, by theorem 5.3.

Example 5.6

The line $3x+4y-12=0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.

Solution

Writing the line $3x+4y=12,$ in intercept form yields $\dfrac{x}{4}+\dfrac{y}{3}=1$. Hence the points A and B are $(4,0)$ and $(0,3)$.

Equation of the circle in diameter form is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$$
$$(x-4)(x-0)+(y-0)(y-3) = 0$$
$$x^2+y^2-4x-3y = 0.$$

Example 5.7

A line $3x+4y+10=0$ cuts a chord of length 6 units on a circle with centre of the circle $(2,1)$. Find the equation of the circle in general form.

Solution

$C(2,1)$ is the centre and $3x+4y+10=0$ cuts a chord AB on the circle. Let M be the midpoint of AB. Then we have

$AM = BM = 3$. Now BMC is a right triangle.

So, we have $\quad CM = \dfrac{|3(2)+4(1)+10|}{\sqrt{3^2+4^2}} = 4.$

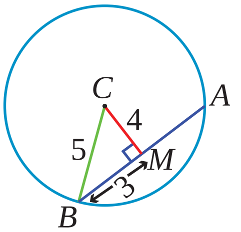

Fig.5.11

By Pythogoras theorem $BC^2 = BM^2+MC^2 = 3^2+4^2 = 25$.

$$BC = 5 = \text{radius}.$$

So, the equation of the required circle is $(x-2)^2+(y-1) = 5^2$

$$x^2+y^2-4x-2y-20 = 0.$$

Example 5.8

A circle of radius 3 units touches both the axes. Find the equations of all possible circles formed in the general form.

Solution

As the circle touches both the axes, the distance of the centre from both the axes is 3 units, centre can be $(\pm 3, \pm 3)$ and hence there are four circles with radius 3, and the required equations of the four circles are $x^2+y^2 \pm 6x \pm 6y + 9 = 0$.

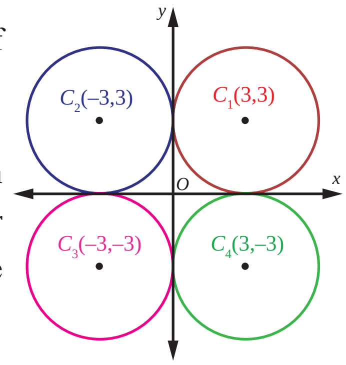

Fig.5.12

Example 5.9

Find the centre and radius of the circle $3x^2+(a+1)y^2+6x-9y+a+4=0$.

Solution

Coefficient of x^2 = Coefficient of y^2 (characteristic (ii) for a second degree equation to represent a circle).

That is, $3=a+1$ and $a=2$.

Therefore, the equation of the circle is

$$3x^2+3y^2+6x-9y+6 = 0$$

$$x^2+y^2+2x-3y+2 = 0$$

So, centre is $\left(-1, \frac{3}{2}\right)$ and radius $r = \sqrt{1+\frac{9}{4}-2} = \frac{\sqrt{5}}{2}$.

Example 5.10

Find the equation of the circle passing through the points $(1,1)$, $(2,-1)$, and $(3,2)$.

Solution

Let the general equation of the circle be

$$x^2+y^2+2gx+2fy+c = 0. \qquad \ldots(1)$$

It passes through points $(1,1), (2,-1)$ and $(3,2)$.

Therefore,

$$2g+2f+c = -2, \qquad \ldots(2)$$

$$4g-2f+c = -5, \qquad \ldots(3)$$

$$6g+4f+c = -13. \qquad \ldots(4)$$

(2) – (3) gives $\quad -2g+4f = 3. \qquad \ldots(5)$

(4) – (3) gives $\quad 2g+6f = -8. \qquad \ldots(6)$

(5) + (6) gives $\quad f = -\frac{1}{2}$

Substituting $f = -\frac{1}{2}$ in (6), $g = -\frac{5}{2}$.

Substituting $f = -\frac{1}{2}$ and $g = -\frac{5}{2}$ in (2), $c = 4$.

Therefore, the required equation of the circle is

$$x^2 + y^2 + 2\left(-\frac{5}{2}\right)x + 2\left(-\frac{1}{2}\right)y + 4 = 0$$

$$\Rightarrow x^2 + y^2 - 5x - y + 4 = 0.$$

Note

Three points on a circle determine the equation to the circle uniquely. Conversely three equidistant points from a centre point forms a circle.

5.2.2 Equations of tangent and normal at a point P on a given circle

Tangent of a circle is a line which touches the circle at only one point and normal is a line perpendicular to the tangent and passing through the point of contact.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Therefore,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \quad \text{... (1)}$$

and $$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \quad \text{,,, (2)}$$

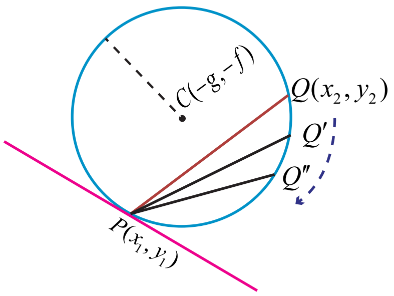

Fig.5.13

$(2)-(1)$ gives

$$x_2^2 - x_1^2 + y_2^2 - y_1^2 + 2g(x_2 - x_1) + 2f(y_2 - y_1) = 0$$

$$(x_2 - x_1)(x_2 + x_1 + 2g) + (y_2 - y_1)(y_2 + y_1 + 2f) = 0$$

$$\frac{x_2 + x_1 + 2g}{y_2 + y_1 + 2f} = -\frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Therefore, slope of $PQ = -\frac{(x_1 + x_2 + 2g)}{(y_1 + y_2 + 2f)}$.

When $Q \to P$, the chord PQ becomes tangent at P

Slope of tangent is $-\frac{(2x_1 + 2g)}{(2y_1 + 2f)} = -\frac{(x_1 + g)}{(y_1 + f)}$.

Hence, the equation of tangent is $y - y_1 = -\frac{(x_1 + g)}{(y_1 + f)}(x - x_1)$. Simplifying,

$$yy_1 + fy - y_1^2 - fy_1 + xx_1 - x_1^2 + gx - gx_1 = 0$$

$$xx_1 + yy_1 + gx + fy - (x_1^2 + y_1^2 + gx_1 + fy_1) = 0 \quad \text{...(1)}$$

Since (x_1, y_1) is a point on the circle, we have $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

Therefore, $-(x_1^2 + y_1^2 + gx_1 + fy_1) = gx_1 + fy_1 + c$. ...(2)

Hence, substituting (2) in (1), we get the equation of tangent at (x_1, y_1) as

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Hence, the equation of normal is $(y-y_1) = \dfrac{(y_1+f)}{(x_1+g)}(x-x_1)$

$$\Rightarrow \quad (y-y_1)(x_1+g) = (y_1+f)(x-x_1)$$

$$\Rightarrow \quad x_1(y-y_1)+g(y-y_1) = y_1(x-x_1)+f(x-x_1)$$

$$\Rightarrow \quad yx_1 - xy_1 + g(y-y_1) - f(x-x_1) = 0.$$

Remark

(1) The equation of tangent at (x_1, y_1) to the circle $x^2+y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(2) The equation of normal at (x_1, y_1) to the circle $x^2+y^2 = a^2$ is $xy_1 - yx_1 = 0$.

(3) The normal passes through the centre of the circle.

5.2.3 Condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ and finding the point of contact

Let the line $y = mx + c$ touch the circle $x^2 + y^2 = a^2$. The centre and radius of the circle $x^2 + y^2 = a^2$ are $(0,0)$ and a respectively.

(i) Condition for a line to be tangent

Then the perpendicular distance of the line $y - mx - c = 0$ from $(0,0)$ is

$$\left|\frac{0-m.0-c}{\sqrt{1+m^2}}\right| = \frac{|c|}{\sqrt{1+m^2}}.$$

This must be equal to radius .Therefore $\dfrac{|c|}{\sqrt{1+m^2}} = a$ or $c^2 = a^2(1+m^2)$.

Thus the condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1+m^2)$.

(ii) Point of contact

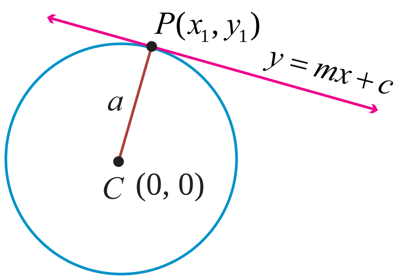

Fig.5.14

Let (x_1, y_1) be the the point of contact of $y = mx + c$ with the circle $x^2 + y^2 = a^2$,

Then $y_1 = mx_1 + c$. ... (1)

Equation of tangent at (x_1, y_1) is $xx_1 + yy_1 = a^2$.

$yy_1 = -xx_1 + a^2$ ... (2)

Equations (1) and (2) represent the same line and hence the coefficients are proportional.

$$\text{So,} \quad \frac{y_1}{1} = \frac{-x_1}{m} = \frac{a^2}{c}$$

$$y_1 = \frac{a^2}{c}, x_1 = \frac{-a^2 m}{c}, \; c = \pm a\sqrt{1+m^2}.$$

Then the points of contact is either $\left(\dfrac{-am}{\sqrt{1+m^2}}, \dfrac{a}{\sqrt{1+m^2}}\right)$

or $\left(\dfrac{am}{\sqrt{1+m^2}}, \dfrac{-a}{\sqrt{1+m^2}}\right)$.

Note

The equation of tangent at P to a circle is $y = mx \pm a\sqrt{1+m^2}$.

Theorem 5.4

From any point outside the circle $x^2 + y^2 = a^2$ two tangents can be drawn.

Proof

Let $P(x_1, y_1)$ be a point outside the circle. The equation of the tangent is $y = mx \pm a\sqrt{1+m^2}$. It passes through (x_1, y_1) . Therefore

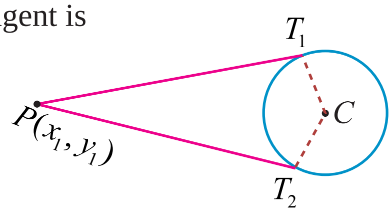

Fig.5.15

$$y_1 = mx_1 \pm a\sqrt{1+m^2}$$

$$y_1 - mx_1 = a\sqrt{1+m^2}$$. Squaring both sides, we get

$$(y_1 - mx_1)^2 = a^2(1+m^2)$$

$$y_1^2 + m^2x_1^2 - 2mx_1y_1 - a^2 - a^2m^2 = 0$$

$$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - a^2) = 0 .$$

This quadratic equation in m gives two values for m .
These values give two tangents to the circle $x^2 + y^2 = a^2$.

Note

(1) If (x_1, y_1) is a point outside the circle, then both the tangents are real.

(2) If (x_1, y_1) is a point inside the circle, then both the tangents are imaginary.

(3) If (x_1, y_1) is a point on the circle, then both the tangents coincide.

Example 5.11

Find the equations of the tangent and normal to the circle $x^2 + y^2 = 25$ at $P(-3, 4)$.

Solution

Equation of tangent to the circle at $P(x_1, y_1)$ is $xx_1 + yy_1 = a^2$.

That is, $$x(-3) + y(4) = 25$$

$$-3x + 4y = 25$$

Equation of normal is $$xy_1 - yx_1 = 0$$

That is, $$4x + 3y = 0 .$$

Example 5.12

If $y = 4x + c$ is a tangent to the circle $x^2 + y^2 = 9$, find c .

Solution

The condition for the line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1+m^2)$.

Then, $$c = \pm\sqrt{9(1+16)}$$

$$c = \pm 3\sqrt{17} .$$

Example 5.13

A road bridge over an irrigation canal have two semi circular vents each with a span of $20m$ and the supporting pillars of width $2m$. Use Fig.5.16 to write the equations that represent the semi-verticular vents.

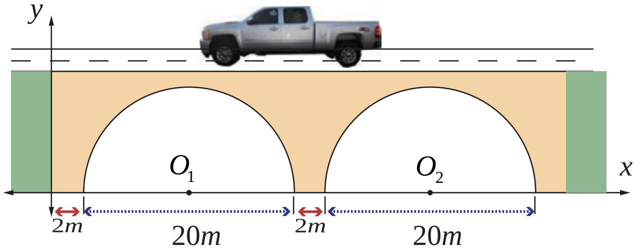

Fig.5.16

Solution

Let O_1 O_2 be the centres of the two semi circular vents.

First vent with centre O_1 $(12,0)$ and radius $r=10$ yields equation to first semicircle as

$$(x-12)^2+(y-0)^2 = 10^2$$

$$\Rightarrow x^2+y^2-24x+44 = 0\,,\, y>0\,.$$

Second vent with centre O_2 $(34,0)$ and radius $r=10$ yields equation to second vent as

$$(x-34)^2+y^2 = 10^2$$

$$\Rightarrow x^2+y^2-68x+1056 = 0\,,\; y>0\,.$$

EXERCISE 5.1

1. Obtain the equation of the circles with radius $5\,\text{cm}$ and touching x-axis at the origin in general form.
2. Find the equation of the circle with centre $(2,-1)$ and passing through the point $(3,6)$ in standard form.
3. Find the equation of circles that touch both the axes and pass through $(-4,-2)$ in general form.
4. Find the equation of the circle with centre $(2,3)$ and passing through the intersection of the lines $3x-2y-1=0$ and $4x+y-27=0$.
5. Obtain the equation of the circle for which $(3,4)$ and $(2,-7)$ are the ends of a diameter.
6. Find the equation of the circle through the points $(1,0)$, $(-1,0)$, and $(0,1)$.
7. A circle of area 9π square units has two of its diameters along the lines $x+y=5$ and $x-y=1$. Find the equation of the circle.
8. If $y=2\sqrt{2}x+c$ is a tangent to the circle $x^2+y^2=16$, find the value of c.
9. Find the equation of the tangent and normal to the circle $x^2+y^2-6x+6y-8=0$ at $(2,2)$.
10. Determine whether the points $(-2,1),(0,0)$ and $(-4,-3)$ lie outside, on or inside the circle $x^2+y^2-5x+2y-5=0$.
11. Find centre and radius of the following circles.
    (i) $x^2+(y+2)^2=0$ (ii) $x^2+y^2+6x-4y+4=0$
    (iii) $x^2+y^2-x+2y-3=0$ (iv) $2x^2+2y^2-6x+4y+2=0$
12. If the equation $3x^2+(3-p)xy+qy^2-2px=8pq$ represents a circle, find p and q. Also determine the centre and radius of the circle.

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5.3. Conics

Definition 5.2

A conic is the locus of a point which moves in a plane, so that its distance from a fixed point bears a constant ratio to its distance from a fixed line not containing the fixed point.

The fixed point is called **focus,** the fixed line is called **directrix** and the constant ratio is called **eccentricity**, which is denoted by ***e***.

(i) If this constant $e=1$ then the conic is called a parabola

(ii) If this constant $e<1$ then the conic is called a ellipse

(iii) If this constant $e>1$ then the conic is called a hyperbola

5.3.1 The general equation of a Conic

Let $S(x_1, y_1)$ be the focus, l the directrix, and e be the eccentricity. Let $P(x, y)$ be the moving point.

By the definition of conic, we have

$$\frac{SP}{PM} = \text{constant } = e, \qquad \ldots(1)$$

Where $SP = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

and PM = perpendicular distance from $P(x, y)$ to the line $lx + my + n = 0$

$$= \left|\frac{lx+my+n}{\sqrt{l^2+m^2}}\right|.$$

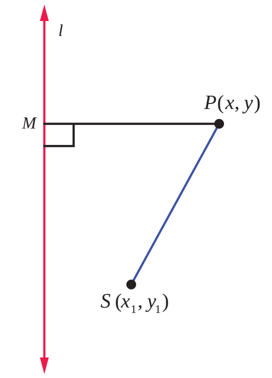


Fig.5.17

From (1) we get $SP^2 = e^2 PM^2$

$$\Rightarrow \quad (x-x_1)^2 + (y-y_1)^2 = e^2\left[\frac{lx+my+n}{\sqrt{l^2+m^2}}\right]^2.$$

On simplification the above equation takes the form of general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where

$$A = 1 - \frac{e^2 l^2}{l^2+m^2}, B = \frac{2lme^2}{l^2+m^2}, C = 1 - \frac{e^2 m^2}{l^2+m^2}$$

Now ,

$$B^2 - 4AC = \frac{4l^2m^2e^4}{\left(l^2+m^2\right)^2} - 4\left(1 - \frac{e^2l^2}{l^2+m^2}\right)\left(1 - \frac{e^2m^2}{l^2+m^2}\right)$$

$$= 4\left(e^2 - 1\right)$$

yielding the following cases:

(i) $B^2 - 4AC = 0 \quad \Leftrightarrow \quad e = 1 \Leftrightarrow$ the conic is a parabola,

(ii) $B^2 - 4AC < 0 \quad \Leftrightarrow \quad 0 < e < 1 \Leftrightarrow$ the conic is an ellipse,

(iii) $B^2 - 4AC > 0 \quad \Leftrightarrow \quad e > 1 \Leftrightarrow$ the conic is a hyperbola.

5.3.2 Parabola

Since $e = 1$, for a parabola, we note that the parabola is the locus of points in a plane that are equidistant from both the directrix and the focus.

(i) Equation of a parabola in standard form with vertex at (0, 0)

Let S be the focus and l be the directrix.

Draw SZ perpendicular to the line l.

Let us assume SZ produced as x-axis and the perpendicular bisector of SZ produced as y- axis. The intersection of this perpendicular bisector with SZ be the origin O.

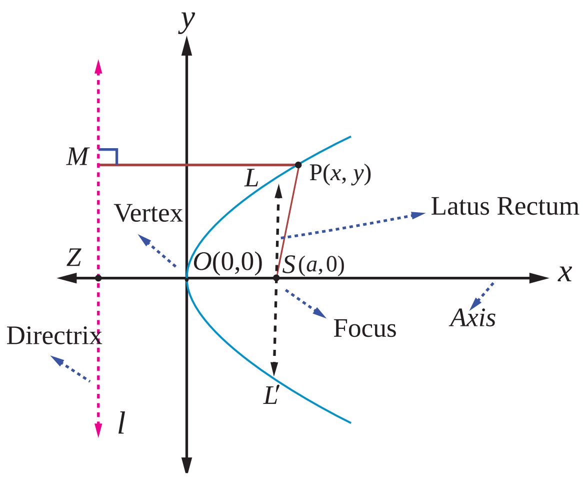


Fig.5.18

Let $SZ = 2a$. Then S is $(a,0)$ and the equation of the directrix is $x + a = 0$.

Let $P(x, y)$ be the moving point in the locus that yield a parabola. Draw PM perpendicular to the directrix. By definition, $e = \frac{SP}{PM} = 1$. So, $SP^2 = PM^2$.

Then, $(x-a)^2 + y^2 = (x+a)^2$. On simplifying, we get $y^2 = 4ax$ which is the equation of the **parabola in the standard form**.

The other standard forms of parabola are $y^2 = -4ax, x^2 = 4ay$, and $x^2 = -4ay$.

Definition 5.3

- The line perpendicular to the directrix and passing through the focus is known as the **Axis** of the parabola.
- The intersection point of the axis with the curve is called **vertex** of the parabola
- Any chord of the parabola, through its focus is called **focal chord** of the parabola
- The length of the focal chord perpendicular to the axis is called **latus rectum** of the parabola

Example 5.14

Find the length of Latus rectum of the parabola $y^2 = 4ax$.

Solution

Equation of the parabola is $y^2 = 4ax$.

Latus rectum LL' passes through the focus $(a,0)$. Refer (Fig. 5.18)

Hence the point L is (a, y_1).

Therefore $y_1^2 = 4a^2$.

Hence $y_1 = \pm 2a$.

The end points of latus rectum are $(a, 2a)$ and $(a, -2a)$.

Therefore length of the latus rectum $LL' = 4a$.

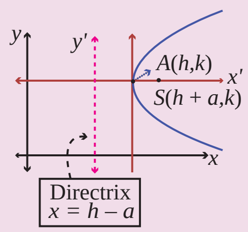
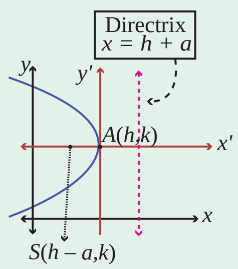
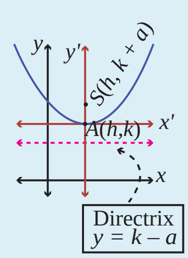
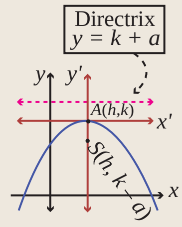
Note

The standard form of the parabola $y^2 = 4ax$ has for its vertex $(0,0)$, axis as x-axis, focus as $(a,0)$. The parabola $y^2 = 4ax$ lies completely on the non-negative side of the x-axis. Replacing y by $-y$ in $y^2 = 4ax$, the equation remains the same. so the parabola $y^2 = 4ax$ is symmetric about x-axis; that is, x-axis is the axis and symmetry of $y^2 = 4ax$

(ii) Parabolas with vertex at (h, k)

When the vertex is (h, k) and the axis of symmetry is parallel to x-axis, the equation of the parabola is either $(y-k)^2 = 4a(x-h)$ or $(y-k)^2 = -4a(x-h)$ (Fig. 5.19, 5.20).

When the vertex is (h, k) and the axis of symmetry is parallel to y-axis, the equation of the parabola is either $(x-h)^2 = 4a(y-k)$ or $(x-h)^2 = -4a(y-k)$ (Fig. 5.21, 5.22).

Equation	Graph	Vertices	Focus	Axis of symmetry	Equation of directrix	Length of latus rectum
$(y-k)^2 = 4a(x-h)$	 (a) The graph of $(y-k)^2 = 4a(x-h)$ Fig. 5.19	(h,k)	$(h+a, 0+k)$	$y=k$	$x=h-a$	$4a$
$(y-k)^2 = -4a(x-h)$	 (b) The graph of $(y-k)^2 = -4a(x-h)$ Fig. 5.20	(h,k)	$(h-a, 0+k)$	$y=k$	$x=h+a$	$4a$
$(x-h)^2 = 4a(y-k)$	 (c) The graph of $(x-h)^2 = 4a(y-k)$ Fig. 5.21	(h,k)	$(0+h, a+k)$	$x=h$	$y=k-a$	$4a$
$(x-h)^2 = -4a(y-k)$	 (d) The graph of $(x-h)^2 = -4a(y-k)$ Fig. 5.22	(h,k)	$(0+h, -a+k)$	$x=h$	$y=k+a$	$4a$

5.3.3 Ellipse

We invoke that an ellipse is the locus of a point which moves such that its distance from a fixed point (focus) bears a constant ratio (eccentricity) less than unity its distance from its directrix bearing a constant ratio e $(0 < e < 1)$.

(i) Equation of an Ellipse in standard form

Let S be a focus, l be a directrix, e be the eccentricity $(0 < e < 1)$ and $P(x, y)$ be the moving point. Draw SZ and PM perpendicular to l.

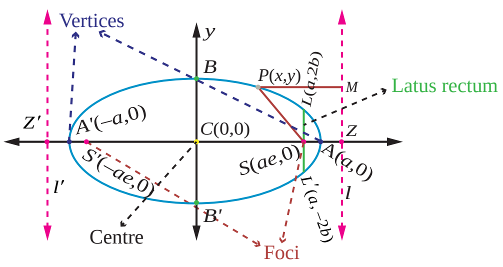


Fig.5.23

Let A and A' be the points which divide SZ internally and externally in the ratio $e:1$ respectively. Let $AA' = 2a$. Let the point of intersection of the perpendicular bisector with AA' be C. Therefore $CA = a$ and $CA' = a$. Choose C as origin and CZ produced as x-axis and the perpendicular bisector of AA' produced as y-axis.

By definition,

$$\frac{SA}{AZ} = \frac{e}{1} \qquad \text{and} \qquad \frac{SA'}{A'Z} = \frac{e}{1}$$

$$SA = eAZ \qquad\qquad SA' = eA'Z$$

$$CA - CS = e(CZ - CA) \qquad\qquad A'C + CS = e(A'C + CZ)$$

$$a - CS = e(CZ - a) \quad \ldots (1) \qquad\qquad a + CS = e(a + CZ) \quad \ldots (2)$$

$(2)+(1)$ gives $CZ = \frac{a}{e}$ and $(2)-(1)$ gives $CS = ae$.

Therefore M is $\left(\frac{a}{e}, y\right)$ and S is $(ae, 0)$.

$$\text{By the definition of a conic, } \frac{SP}{PM} = e \Rightarrow SP^2 = e^2 PM^2$$

$$\Rightarrow (x - ae)^2 + (y - 0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + 0\right] \text{ which}$$

$$\text{on simplification yields } \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

$$\text{Since } 1 - e^2 \text{ is a positive quantity, write } b^2 = a^2(1 - e^2)$$

$$\text{Taking } ae = c,\ b^2 = a^2 - c^2.$$

Hence we obtain the locus of P as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is the equation of an **ellipse in standard form** and note that it is symmetrical about x and y axis.

Definition 5.4

(1) The line segment AA' is called the **major axis** of the ellipse and is of length $2a$.

(2) The line segment BB' is called the **minor axis** of the ellipse and is of length $2b$.

(3) The line segment $CA =$ the line segment $CA' =$ **semi major axis** $= \boldsymbol{a}$ and the line segment $CB =$ the line segment $CB' =$ **semi minor axis** $= \boldsymbol{b}$.

(4) By symmetry, taking $S'(-ae, 0)$ as focus and $x = -\frac{a}{e}$ as directrix l' gives the same ellipse.

Thus, we see that an ellipse has two foci, $S(ae, 0)$ and $S'(-ae, 0)$ and two vertices $A(a, 0)$ and $A'(-a, 0)$ and also two directrices, $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

Example 5.15

Find the length of Latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

The Latus rectum LL' (Fig. 5.22) of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through $S(ae,0)$.

Hence L is (ae, y_1).

Therefore,
$$\frac{a^2e^2}{a^2}+\frac{y_1^{\,2}}{b^2}=1$$
$$\frac{y_1^{\,2}}{b^2}=1-e^2$$
$$y_1^{\,2}=b^2(1-e^2)$$
$$=b^2\left(\frac{b^2}{a^2}\right)\quad\left(\text{since, } e^2=1-\frac{b^2}{a^2}\right)$$
$$y_1=\pm\frac{b^2}{a}.$$

That is, the end points of Latus rectum L and L' are $\left(ae,\frac{b^2}{a}\right)$ and $\left(ae,-\frac{b^2}{a}\right)$.

Hence the length of latus rectum $LL'=\frac{2b^2}{a}$.

(ii) Types of ellipses with centre at (h,k)

(a) Major axis parallel to the x-axis

From Fig. 5.24
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a>b$$

The length of the major axis is $2a$. The length of the minor axis is $2b$. The coordinates of the vertices are $(h+a,k)$ and $(h-a,k)$, and the coordinates of the foci are $(h+c,k)$ and $(h-c,k)$ where $c^2=a^2-b^2$.

(b) Major axis parallel to the y-axis

From Fig. 5.25
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a>b$$

The length of the major axis is $2a$. The length of the minor axis is $2b$. The coordinates of the vertices are $(h,k+a)$ and $(h,k-a)$, and the coordinates of the foci are $(h,k+c)$ and $(h,k-c)$, where $c^2=a^2-b^2$.

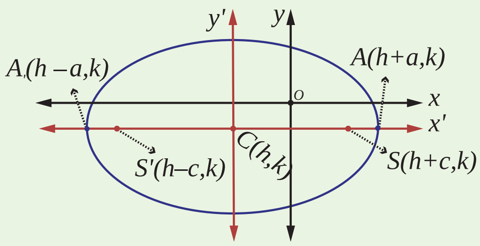
<table>
<tr><th>Equation</th><th>Centre</th><th>Major Axis</th><th>Vertices</th><th>Foci</th></tr>
<tr><td>$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1 \quad \boxed{a^2 > b^2}$


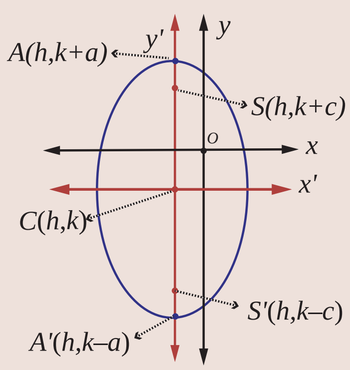
Fig.5.24
(a) Major axis parallel to the x-axis
Foci are c units right and c units left of centre, where $c^2 = a^2 - b^2$.</td><td>(h, k)</td><td>parallel to the x-axis</td><td>$(h-a, k)$
$(h+a, k)$</td><td>$(h-c, k)$
$(h+c, k)$</td></tr>
<tr><td>$\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1 \quad \boxed{a^2 > b^2}$


Fig.5.25
(b) Major axis parallel to the y-axis
Foci are c units right and c units left of centre, where $c^2 = a^2 - b^2$.</td><td>(h, k)</td><td>parallel to the y-axis</td><td>$(h, k-a)$
$(h, k+a)$</td><td>$(h, k-c)$
$(h, k+c)$</td></tr>
</table>

Theorem 5.5

The sum of the focal distances of any point on the ellipse is equal to length of the major axis.

Proof

Let $P(x, y)$ be a point on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Draw MM' through P, perpendicular to directrices l and l'.

Draw $PN \perp$ to x-axis.

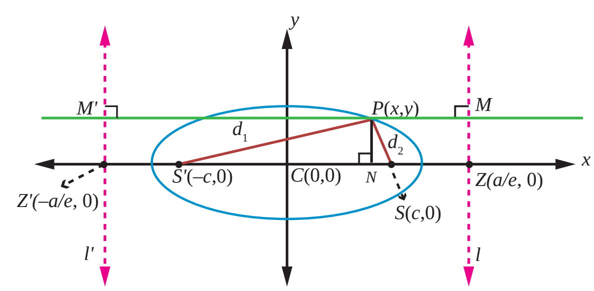


Fig.5.26

By definition

$$\begin{aligned} SP &= ePM \\ &= eNZ \\ &= e[CZ - CN] \end{aligned}$$

$$= e\left[\frac{a}{2} - x\right] = a - ex \qquad \text{... (1)}$$

and
$$\begin{aligned} SP' &= ePM' \\ &= e[CN + CZ'] \\ &= e\left[x + \frac{a}{e}\right] = ex + a \qquad \text{... (2)} \end{aligned}$$

Hence, $SP + S'P = a - ex + a + ex = 2a$

Remark

When $b = a$, the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, becomes $(x-h)^2 + (y-k)^2 = a^2$ the equation of circle with centre (h,k) and radius a.

When $b = a, e = \sqrt{1 - \frac{a^2}{a^2}} = 0$. **Hence the eccentricity of the circle is zero.**

Furthere, $\frac{SP}{PM} = 0$ implies $PM \to \infty$. That is, the directrix of the circle is at infinity.

Remark

Auxiliary circle or circumcircle is the circle with length of major axis as diameter and **Incircle** is the circle with length of minor axis as diameter. They are given by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ respectively.

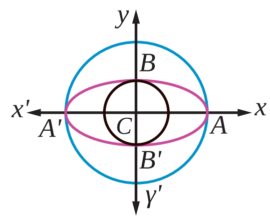


Fig.5.27

5.3.4 Hyperbola

We invoke that a hyperbola is the locus of a point which moves such that its distance from a fixed point (focus) bears a constant ratio (eccentricity) greater than unity its distance from its directrix, bearing a constant ratio e $(e > 1)$.

(i) Equation of a Hyperbola in standard form with centre at (0, 0)

Let S be a focus, l be the directrix line, e be the eccentricity $e > 1$ and $P(x, y)$ be the moving point. Draw SZ and PM perpendicular to l.

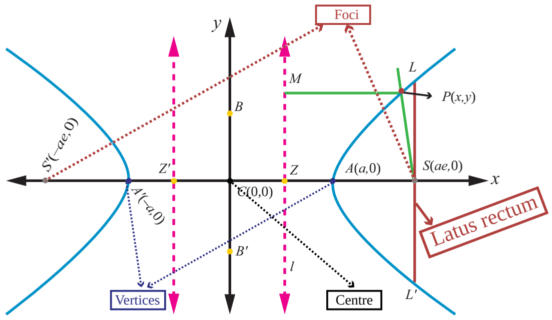


Fig.5.28

Let A and A' be the points which divide SZ internally and externally in the ratio $e:1$ respectively.

Let $AA' = 2a$. Let the point of intersection of the perpendicular bisector with AA' be C. Then CA = CA′= a Choose C as origin and the line CZ produced as x-axis and the perpendicular bisector of AA' as y-axis.

By definition, $\frac{AS}{AZ} = e$ and $\frac{A'S}{A'Z} = e$.

$\Rightarrow \quad AS = eAZ \qquad\qquad A'S = eA'Z$

$\Rightarrow \quad CS - CA = e(CA - CZ) \qquad\qquad A'C + CS = e(A'C + CZ)$

$\Rightarrow \quad CS - a = e(a - CZ) \ldots (1) \qquad\qquad a + CS = e(a + CZ) \ldots (2)$

$(1)+(2)$ gives $CS = ae$ and $(2)-(1)$ gives $CZ = \frac{a}{e}$.

Hence, the coordinates of S are $(ae, 0)$. Since $PM = x - \frac{a}{e}$, the equation of directrix is $x - \frac{a}{e} = 0$.

Let $P(x, y)$ be any point on the hyperbola.

By the definition of a conic, $\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 PM^2$.

Then $(x - ae)^2 + (y - 0)^2 = e^2\left(x - \frac{a}{e}\right)^2$

$\Rightarrow \quad (x - ae)^2 + y^2 = (ex - a)^2$

$\Rightarrow \quad (e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$

$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$. Since $e > 1, a^2\left(e^2 - 1\right) > 0$. Setting $a^2(e^2 - 1) = b^2$, we obtain the locus of P as

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which is the equation of a **Hyperbola in standard form** and note that it is symmetrical about x and y-axes.

Taking $ae = c$, we get $b^2 = c^2 - a^2$.

Definition 5.5

(1) The line segment AA' is the **transverse** axis of length $2a$.

(2) The line segment BB' is the **conjugate** axis of length $2b$.

(3) The line segment $CA =$ the line segment $CA' =$ **semi transverse axis** $= \boldsymbol{a}$ and the line segment $CB =$ the line segment $CB' =$ **semi conjugate axis** $= \boldsymbol{b}$.

(4) By symmetry, taking $S'(-ae, 0)$ as focus and $x = -\frac{a}{e}$ as directrix l' gives the same hyperbola.

Thus we see that a hyperbola has two foci $S(ae, 0)$ and $S'(-ae, 0)$, two vertices $A(a, 0)$ and $A'(-a, 0)$ and two directrices $x = \frac{a}{e}$ and $x = -\frac{a}{e}$.

Length of latus rectum of hyperbola is $\frac{2b^2}{a}$,which can be obtained along lines as that of the ellipse.

Asymptotes

Let $P(x, y)$ be a point on the curve defined by $y = f(x)$, which moves further and further away from the origin such that the distance between P and some fixed line tends to zero. This fixed line is called an **asymptote**.

Note that the hyperbolas admit asymptotes while parabolas and ellipses do not.

(ii) Types of Hyperbola with centre at (h, k)

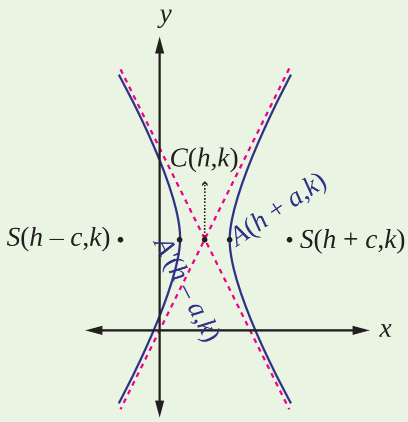


Fig. 5.29
(a) transverse axis parallel to the *x*-axis

(a) Transverse axis parallel to the *x*-axis.

The equation of a hyperbola with centre $C(h,k)$ and transverse axis parallel to the *x*-axis (Fig. 5.29) is given by $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$.

The coordinates of the vertices are $A(h+a,k)$ and $A'(h-a,k)$. The coordinates of the foci are $S(h+c,k)$ and $S'(h-c,k)$ where $c^2=a^2+b^2$.

The equations of directrices are $x=h\pm\frac{a}{e}$.

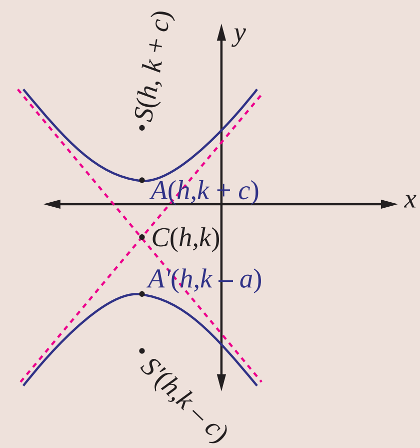


Fig. 5.30
(b) transverse axis parallel to the *y*-axis

(b) Transverse axis parallel to the *y*-axis

The equation of a hyperbola with centre $C(h,k)$ and transverse axis parallel to the y-axis (Fig. 5.0) is given by

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$$

The coordinates of the vertices are $A(h,k+a)$ and $A'(h,k-a)$.The coordinates of the foci are $S(h,k+c)$ and $S'(h,k-c)$, where $c^2=a^2+b^2$.

The equations of directrices are $y=k\pm\frac{a}{e}$.

Remark

(1) The circle described on the transverse axis of hyperbola as its diameter is called the auxiliary circle of the hyperbola. Its equation is $x^2+y^2=a^2$.

(2) The absolute difference of the focal distances of any point on the hyperbola is constant and is equal to length of transverse axis. That is, $|PS-PS'|=2a$. (can be proved similar that of ellipse)

So far we have discussed four standard types of parabolas, two types of ellipses and two types of hyperbolas. There are plenty of parabolas, ellipses and hyperbolas whose equations cannot be classified under the standard types, For instance consider the following parabola, ellipse, and hyperbola.

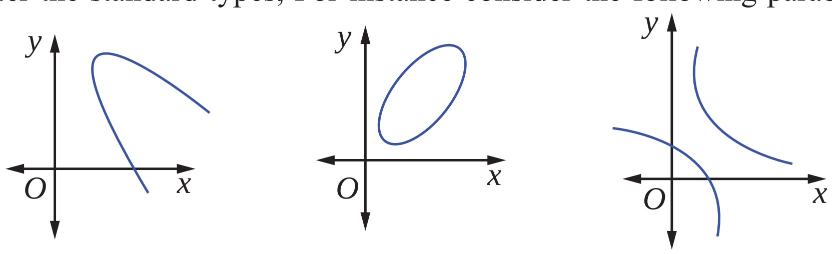


Fig. 5.31

By a suitable transformation of coordinate axes they can be represented by standard equations.

Example 5.16

Find the equation of the parabola with focus $\left(-\sqrt{2},0\right)$ and directrix $x=\sqrt{2}$.

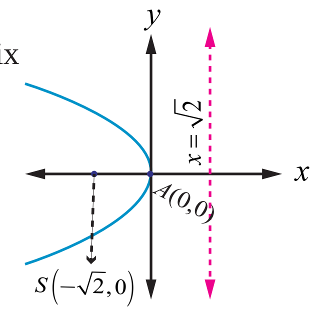


Fig.5.32

Solution

Parabola is open left and axis of symmetry as x-axis and vertex $(0,0)$.

Then the equation of the required parabola is

$$\left(y-0\right)^2 = -4\sqrt{2}\left(x-0\right)$$

$$\Rightarrow \quad y^2 = -4\sqrt{2}\,x.$$

Example 5.17

Find the equation of the parabola whose vertex is $(5,-2)$ and focus $(2,-2)$.

Solution

Given vertex $A(5,-2)$ and focus $S(2,-2)$ and the focal distance

$AS=a=3$.

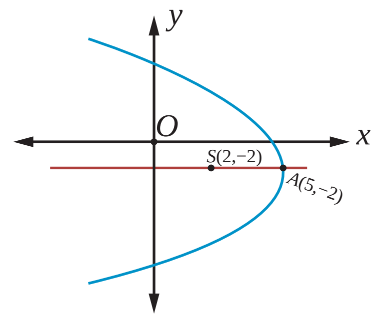


Fig.5.33

Parabola is open left and symmetric about the line parallel to x-axis.

Then, the equation of the required parabola is

$$\left(y+2\right)^2 = -4\left(3\right)\left(x-5\right)$$

$$\Rightarrow \quad y^2+4y+4 = -12x+60$$

$$\Rightarrow \quad y^2+4y+12x-56=0.$$

Example 5.18

Find the equation of the parabola with vertex $(-1,-2)$, axis parallel to y-axis and passing through $(3,6)$.

Solution

Since axis is parallel to y-axis the required equation of the parabola is

$$\left(x+1\right)^2 = 4a\left(y+2\right).$$

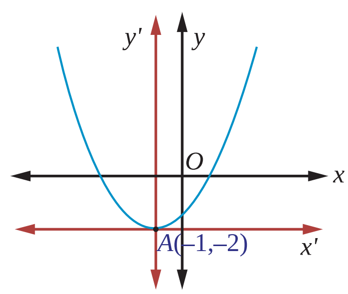


Fig.5.34

Since this passes through (3,6), we get

$$\left(3+1\right)^2 = 4a\left(6+2\right)$$

$$\Rightarrow \quad a = \frac{1}{2}.$$

Then the equation of parabola is $\left(x+1\right)^2 = 2\left(y+2\right)$ which on simplifying yields,

$$x^2+2x-2y-3 = 0.$$

Example 5.19

Find the vertex, focus, directrix, and length of the latus rectum of the parabola $x^2-4x-5y-1=0$.

Solution

For the parabola,

$$x^2-4x-5y-1 = 0$$

$$\Rightarrow \quad x^2-4x = 5y+1$$

$$\Rightarrow \quad x^2-4x+4 = 5y+1+4.$$

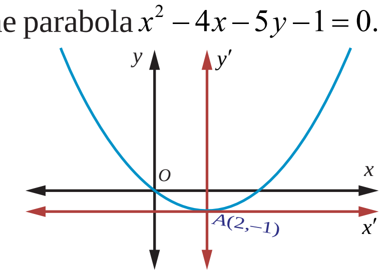


Fig.5.35

$\Rightarrow \quad (x-2)^2 = 5(y+1)$ which is in standard form. Therefore, $4a = 5$ and the vertex is $(2,-1)$, and the focus is $\left(2,\frac{1}{4}\right)$.

Equation of directrix is
$$y-k+a = 0$$
$$y+1+\frac{5}{4} = 0$$
$$4y+9 = 0.$$

Length of latus rectum 5 units.

Example 5.20

Find the equation of the ellipse with foci $(\pm 2,0)$, vertices $(\pm 3,0)$.

Solution

From Fig. 5.36, we get
$$SS' = 2c \text{ and } 2c = 4 \quad ; A'A = 2a = 6$$
$$\Rightarrow \quad c = 2 \text{ and } a = 3,$$
$$\Rightarrow b^2 = a^2 - c^2 = 9-4 = 5.$$

Major axis is along x-axis, since $a > b$.

Centre is (0, 0) and Foci are $(\pm 2,0)$.

Therefore, equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$.

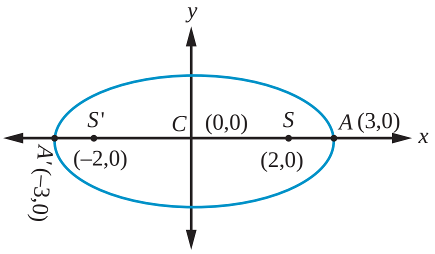


Fig.5.36

Example 5.21

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is $(2,3)$ and a directrix is $x=7$. Also find the length of the major and minor axes of the ellipse.

Solution

By the definition of a conic, $\frac{SP}{PM} = e$ or $SP^2 = e^2PM^2$.

Then,
$$(x-2)^2+(y-3)^2 = \frac{1}{4}(x-7)^2$$
$$\Rightarrow \quad 3x^2+4y^2-2x-24y+3 = 0$$
$$\Rightarrow \quad 3\left(x-\frac{1}{3}\right)^2+4(y-3)^2 = 3\left(\frac{1}{9}\right)+4\times 9-3 = \frac{100}{3}$$
$$\Rightarrow \quad \frac{\left(x-\frac{1}{3}\right)^2}{\frac{100}{9}}+\frac{(y-3)^2}{\frac{100}{12}} = 1 \text{ which is in the standard form.}$$

Therefore, the length of major axis $= 2a = 2\sqrt{\frac{100}{9}} = \frac{20}{3}$ and

the length of minor axis $= 2b = 2\sqrt{\frac{100}{12}} = \frac{10}{\sqrt{3}}$.

Example 5.22

Find the foci, vertices and length of major and minor axis of the conic $4x^2+36y^2+40x-288y+532=0$.

Solution

Completing the square on x and y of $4x^2+36y^2+40x-288y+532=0$,

$$4(x^2+10x+25-25)+36(y^2-8y+16-16)+532 = 0 \text{, gives}$$

$$4(x^2+10x+25)+36(y^2-8y+16) = -532+100+576$$

$$4(x+5)^2+36(y-4)^2 = 144.$$

Dividing both sides by 144, the equation reduces to

$$\frac{(x+5)^2}{36}+\frac{(y-4)^2}{4} = 1.$$

This is an ellipse with centre $(-5,4)$, major axis is parallel to x-axis, length of major axis is 12 and length of minor axis is 4. Vertices are $(1,4)$ and $(-11,4)$.

$$\text{Now, } c^2 = a^2-b^2 = 36-4 = 32$$

$$\text{and } c = \pm 4\sqrt{2}.$$

Then the foci are $\left(-5-4\sqrt{2},4\right)$ and $\left(-5+4\sqrt{2},4\right)$.

Length of the major axis = $2a=12$ units and

the length of the minor axis = $2b=4$ units.

Example 5.23

For the ellipse $4x^2+y^2+24x-2y+21=0$, find the centre, vertices, and the foci. Also prove that the length of latus rectum is 2.

Solution

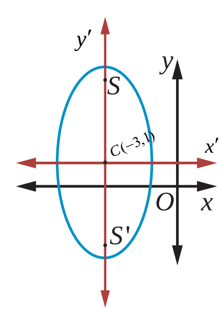


Fig.5.37

Rearranging the terms, the equation of ellipse is

$$4x^2+24x+y^2-2y+21 = 0$$

$$\text{That is, } 4\left(x^2+6x+9-9\right)+\left(y^2-2y+1-1\right)+21 = 0,$$

$$4(x+3)^2-36+(y-1)^2-1+21 = 0,$$

$$4(x+3)^2+(y-1)^2 = 16,$$

$$\frac{(x+3)^2}{4}+\frac{(y-1)^2}{16} = 1.$$

Centre is $(-3,1)$ $a=4$, $b=2$, and the major axis is parallel to y-axis

$$c^2 = 16-4 = 12$$

$$c = \pm 2\sqrt{3}.$$

Therefore, the foci are $\left(-3,2\sqrt{3}+1\right)$ and $\left(-3,-2\sqrt{3}+1\right)$.

Vertices are $(3,\pm 4+1)$. That is the vertices are $(3,5)$ and $(3,-3)$, and

the length of Latus rectum = $\dfrac{2b^2}{a}=2$ units. (see Fig. 5.37)

Example 5.24

Find the equation of the hyperbola with vertices $(0,\pm 4)$ and foci $(0,\pm 6)$.

Solution

From Fig. 5.38, the midpoint of line joining foci is the centre $C\,(0,0)$.

Transverse axis is y-axis

$$AA' = 2a \Rightarrow 2a = 8,$$
$$SS' = 2c = 12, c = 6$$
$$a = 4$$
$$b^2 = c^2 - a^2 = 36 - 16 = 20.$$

Hence the equation of the required hyperbola is $\frac{y^2}{16} - \frac{x^2}{20} = 1.$

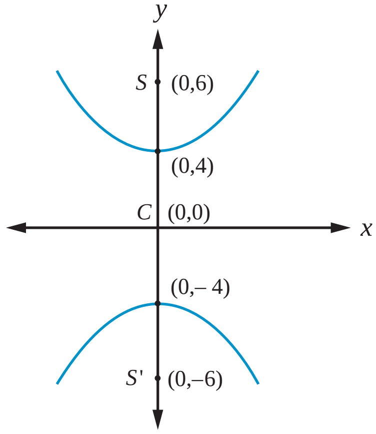


Fig.5.38

Example 5.25

Find the vertices, foci for the hyperbola $9x^2 - 16y^2 = 144$.

Solution

Reducing $9x^2 - 16y^2 = 144$ to the standard form,

we have, $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

With the transverse axis is along x-axis vertices are $(-4,0)$ and $(4,0)$;

and $c^2 = a^2 + b^2 = 16 + 9 = 25$, $c = 5$.

Hence the foci are $(-5,0)$ and $(5,0)$.

Example 5.26

Find the centre, foci, and eccentricity of the hyperbola $11x^2 - 25y^2 - 44x + 50y - 256 = 0$

Solution

Rearranging terms in the equation of hyperbola to bring it to standard form,

we have, $11(x^2 - 4x) - 25(y^2 - 2y) - 256 = 0$

$$11(x-2)^2 - 25(y-1)^2 = 256 - 44 + 25$$
$$11(x-2)^2 - 25(y-1)^2 = 275$$
$$\frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1.$$

Centre $(2,1)$, $a^2 = 25, b^2 = 11$

$$c^2 = a^2 + b^2$$
$$= 25 + 11 = 36$$

Therefore, $c = \pm 6$

and $e = \frac{c}{a} = \frac{6}{5}$ and the coordinates of foci are $(8,1)$ and $(-4,1)$ from Fig. 5.39.

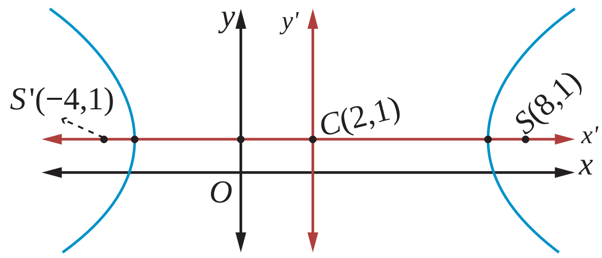


Fig. 5.39

Example 5.27

The orbit of Halley's Comet (Fig. 5.51) is an ellipse 36.18 astronomical units long and by 9.12 astronomical units wide. Find its eccentricity.

Solution

Given that $2a = 36.18,\ 2b = 9.12$, we get

$$e = \sqrt{1-\frac{b^2}{a^2}} = \frac{\sqrt{a^2-b^2}}{a} = \frac{\sqrt{\left(\frac{36.18}{2}\right)^2 - \left(\frac{9.12}{2}\right)^2}}{\frac{36.18}{2}}$$

$$= \frac{\sqrt{(18.09)^2-(4.56)^2}}{(8.09)} \approx 0.97.$$

Note

One astronomical unit (mean distance of Sun and earth) is $1,49,597,870\,km$, the semi major axis of the Earth's orbit.

EXERCISE 5.2

1. Find the equation of the parabola in each of the cases given below:
   (i) focus $(4,0)$ and directrix $x=-4$.
   (ii) passes through $(2,-3)$ and symmetric about y -axis.
   (iii) vertex $(1,-2)$ and focus $(4,-2)$.
   (iv) end points of latus rectum $(4,-8)$ and $(4,8)$.
2. Find the equation of the ellipse in each of the cases given below:
   (i) foci $(\pm 3,0), e=\frac{1}{2}$.
   (ii) foci $(0,\pm 4)$ and end points of major axis are $(0,\pm 5)$.
   (iii) length of latus rectum 8, eccentricity $=\frac{3}{5}$, centre $(0, 0)$ and major axis on x -axis.
   (iv) length of latus rectum 4, distance between foci $4\sqrt{2}$, centre $(0, 0)$ and major axis as y - axis.
3. Find the equation of the hyperbola in each of the cases given below:
   (i) foci $(\pm 2,0)$, eccentricity $=\frac{3}{2}$.
   (ii) Centre $(2,1)$, one of the foci $(8,1)$ and corresponding directrix $x=4$.
   (iii) passing through $(5,-2)$ and length of the transverse axis along x axis and of length 8 units.

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:
   (i) $y^2 = 16x$ (ii) $x^2 = 24y$ (iii) $y^2 = -8x$
   (iv) $x^2 - 2x + 8y + 17 = 0$ (v) $y^2 - 4y - 8x + 12 = 0$

5. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:
   (i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$ (ii) $\frac{x^2}{3} + \frac{y^2}{10} = 1$ (iii) $\frac{x^2}{25} - \frac{y^2}{144} = 1$ (iv) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

6. Prove that the length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

7. Show that the absolute value of difference of the focal distances of any point P on the hyperbola is the length of its transverse axis.

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :
   (i) $\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$ (ii) $\frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$ (iii) $\frac{(x+3)^2}{225} - \frac{(y-4)^2}{64} = 1$
   (iv) $\frac{(y-2)^2}{25} - \frac{(x+1)^2}{16} = 1$ (v) $18x^2 + 12y^2 - 144x + 48y + 120 = 0$
   (vi) $9x^2 - y^2 - 36x - 6y + 18 = 0$

5.4 Conic Sections

In addition to the method to determine the curves discussed in Section 5.3, geometric description of a conic section is given here. The graph of a circle, an ellipse, a parabola, or a hyperbola can be obtained by the intersection of a plane and a double napped cone. Hence, these figures are referred to as conic sections or simply conics.

5.4.1 Geometric description of conic section

A plane perpendicular to the axis of the cone (plane C) intersecting any one nape of the double napped cone yields a circle (Fig. 5.40) . The plane E, tilted so that it is not perpendicular to the axis, intersecting any one nape of the double napped cone yields an ellipse (Fig. 5.40). When the plane is parallel to a side of one napes of the double napped cone, the plane intersecting the cone yields a parabola (Fig. 5.41). When the plane is parallel to the plane containing the axis of the double cone, intersecting the double cone yields a hyperbola (Fig. 5.42).

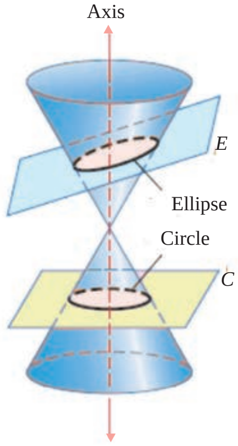


Fig. 5.40

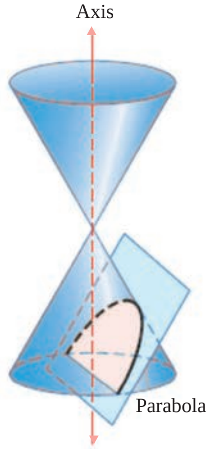


Fig. 5.41

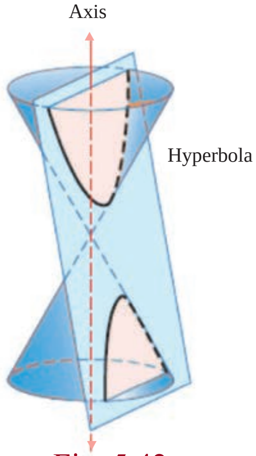


Fig. 5.42

5.4.2 Degenerate Forms

Degenerate forms of various conics (Fig. 5.43) are either a point or a line or a pair of straight lines or two intersecting lines or empty set depending on the angle (nature) of intersection of the plane with the double napped cone and passing through the vertex or when the cones degenerate into a cylinder with the plane parallel to the axis of the cylinder.

If the intersecting plane passes through the vertex of the double napped cone and perpendicular to the axis, then we obtain a point or a point circle. If the intersecting plane passes through a generator then we obtain a line or a pair of parallel lines, a degenerate form of a parabola for which $A = B = C = 0$ in general equation of a conic and if the intersecting plane passes through the axis and passes through the vertex of the double napped cone, then we obtain intersecting lines a degenerate of the hyperbola.

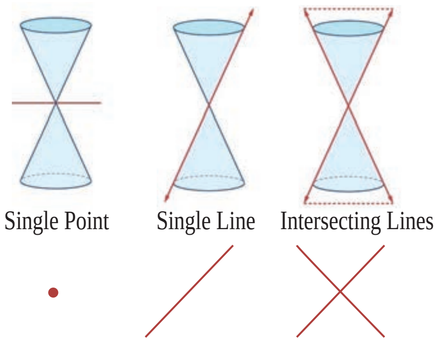


Fig. 5.43

Remark

In the case of an ellipse $(0 < e < 1)$ where $e = \sqrt{1 - \frac{b^2}{a^2}}$. As $e \to 0$, $\frac{b}{a} \to 1$ i.e., $b \to a$ or the lengths of the minor and major axes are close in size. i.e., the ellipse is close to being a circle. As $e \to 1$, $\frac{b}{a} \to 0$ and the ellipse degenerates into a line segment i.e., the ellipse is flat.

Remark

In the case of a hyperbola $(e > 1)$ where $e = \sqrt{1 + \frac{b^2}{a^2}}$. As $e \to 1$, $\frac{b}{a} \to 0$ i.e., as $e \to 1$, b is very small related to a and the hyperbola becomes a pointed nose. As $e \to \infty$, b is very large related to a and the hyperbola becomes flat.

5.4.3 Identifying the conics from the general equation of the conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

The graph of the second degree equation is one of a circle, parabola, an ellipse, a hyperbola, a point, an empty set, a single line or a pair of lines. When,

(1) $A = C = 1,\ B = 0, D = -2h,\ E = -2k,\ F = h^2 + k^2 - r^2$ the general equation reduces to $(x-h)^2 + (y-k)^2 = r^2$, which is a circle.

(2) $B = 0$ and either A or $C = 0$, the general equation yields a parabola under study, at this level.

(3) $A \neq C$ and A and C are of the same sign, the general equation yields an ellipse.

(4) $A \neq C$ and A and C are of opposite signs, the general equation yields a hyperbola

(5) $A = C$ and $B = D = E = F = 0$, the general equation yields a point $x^2 + y^2 = 0$.

(6) $A = C = F$ and $B = D = E = 0$, the general equation yields an empty set $x^2 + y^2 + 1 = 0$, as there is no real solution.

(7) $A \neq 0$ or $C \neq 0$ and others are zeros, the general equation yield coordinate axes.

(8) $A = -C$ and rests are zero, the general equation yields a pair of lines $x^2 - y^2 = 0$.

Example 5.28

Identify the type of the conic for the following equations:

(1) $16y^2 = -4x^2 + 64$ (2) $x^2 + y^2 = -4x - y + 4$

(3) $x^2 - 2y = x + 3$ (4) $4x^2 - 9y^2 - 16x + 18y - 29 = 0$

Solution

Q.no.	Equation	condition	Type of the conic
1	$16y^2 = -4x^2 + 64$	3	Ellipse
2	$x^2 + y^2 = -4x - y + 4$	1	Circle
3	$x^2 - 2y = x + 3$	2	parabola
4	$4x^2 - 9y^2 - 16x + 18y - 29 = 0$	4	Hyperbola

EXERCISE 5.3

Identify the type of conic section for each of the equations.

1. $2x^2 - y^2 = 7$
2. $3x^2 + 3y^2 - 4x + 3y + 10 = 0$
3. $3x^2 + 2y^2 = 14$
4. $x^2 + y^2 + x - y = 0$
5. $11x^2 - 25y^2 - 44x + 50y - 256 = 0$
6. $y^2 + 4x + 3y + 4 = 0$

5.5 Parametric form of Conics

5.5.1 Parametric equations

Suppose $f(t)$ and $g(t)$ are functions of $'t'$. Then the equations $x = f(t)$ and $y = g(t)$ together describe a curve in the plane . In general $'t'$ is simply an arbitrary variable, called in this case a **parameter**, and this method of specifying a curve is known as **parametric equations**. One important interpretation of $'t'$ is time . In this interpretation,the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time $'t'$.

So a parametric equation simply has a third variable, expressing x and y in terms of that third variable as a parameter . A parameter does not always have to be $'t'$. Using $'t'$ is more standard but one can use any other variable.

(i) Parametric form of the circle $x^2 + y^2 = a^2$

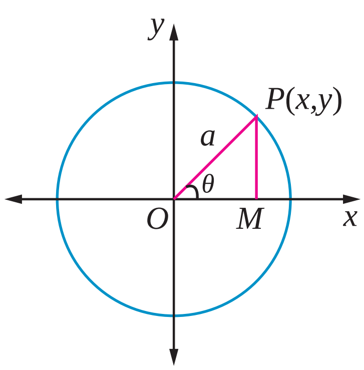


Fig. 5.44

Let $P(x, y)$ be any point on the circle $x^2 + y^2 = a^2$.

Join OP and let it make an angle θ with x-axis.

Draw PM perpendicular to x-axis. From triangle OPM,

$x = OM = a\cos\theta$

$y = MP = a\sin\theta$

Thus the coordinates of any point on the given circle are $(a\cos\theta, a\sin\theta)$ and

$x = a\cos\theta, y = a\sin\theta,\ 0 \le \theta \le 2\pi$ are the parametric equations of the circle $x^2 + y^2 = a^2$.

Conversely, if $x = a\cos\theta,\ \ y = a\sin\theta,\ \ 0 \le \theta \le 2\pi,$

then, $\dfrac{x}{a} = \cos\theta, \dfrac{y}{a} = \sin\theta$.

Squaring and adding, we get,

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = \cos^2\theta + \sin^2\theta = 1.$$

Thus $x^2 + y^2 = a^2$ yields the equation to circle with centre $(0,0)$ and radius a units.

Note

(1) $x = a\cos t, y = a\sin t$, $0 \le t \le 2\pi$ also represents the same parametric equations of circle $x^2 + y^2 = a^2$,

t increasing in anticlockwise direction.

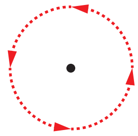


Fig. 5.45

(2) $x = a\sin t, y = a\cos t, 0 \le t \le 2\pi$ also represents the same parametric equations of circle $x^2 + y^2 = a^2$,

t increasing in clockwise direction.

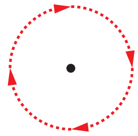


Fig. 5.46

(ii) Parametric form of the parabola $y^2 = 4ax$

Let $P(x_1, y_1)$ be a point on the parabola

$$
\begin{aligned}
y_1^2 &= 4ax_1\\
(y_1)(y_1) &= (2a)(2x_1)\\
\frac{y_1}{2a} &= \frac{2x_1}{y_1} = t \quad (-\infty < t < \infty) \text{ say}\\
y_1 &= 2at,\ 2x_1 = y_1 t\\
2x_1 &= 2at(t)\\
x_1 &= at^2
\end{aligned}
$$

Parametric form of $y^2 = 4ax$ is $x = at^2, y = 2at, -\infty < t < \infty$.

Conversely if $x = at^2$ and $y = 2at, -\infty < t < \infty$, then eliminating $'t'$ between these equations we get $y^2 = 4ax$.

(iii) Parametric form of the Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

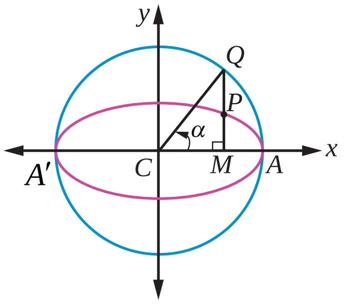


Fig. 5.47

Let P be any point on the ellipse. Let the ordinate MP meet the auxiliary circle at Q.

Let $\angle ACQ = \alpha$

$\therefore\ CM = a\cos\alpha,\ \ MQ = a\sin\alpha$

and $Q(a\cos\alpha, a\sin\alpha)$

Now x-coordinate of P is $a\cos\alpha$. If its y-coordinate is y', then $P(a\cos\alpha, y')$ lies on

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

whence $\cos^2\alpha + \frac{y'^2}{b^2} = 1$

$\Rightarrow$ $y' = b\sin\alpha$.

Hence P is $(a\cos\alpha, b\sin\alpha)$.

The parameter α is called the eccentric angle of the point P. Note that α is the angle which the line CQ makes with the x-axis and not the angle which the line CP makes with it.

Hence the parametric equation of an ellipse is $x = a\cos\theta,\ y = b\sin\theta$, where θ is the parameter $0 \le \theta \le 2\pi$.

(iv) Parametric form of the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Similarly, parametric equation of a hyperbola can be derived as $x = a\sec\theta,\ y = b\tan\theta$, where θ is the parameter. $-\pi \le \theta \le \pi$ except $\theta = \pm\frac{\pi}{2}$.

In nutshell the parametric equations of the circle, parabola,ellipse and hyperbola are given in the following table.

Conic	Parametric equations	Parameter	Range of parameter	Any point on the conic
Circle	$x = a\cos\theta$ $y = a\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, a\sin\theta)$
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\theta$ $y = b\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, b\sin\theta)$
Hyperbola	$x = a\sec\theta$ $y = b\tan\theta$	θ	$-\pi \le \theta \le \pi$ except $\theta = \pm\frac{\pi}{2}$	'θ' or $(a\sec\theta, b\tan\theta)$

Remark

(1) Parametric form represents a family of points on the conic which is the role of a parameter. Further parameter plays the role of a constant and a variable, while cartesian form represents the locus of a point describing the conic. Parameterisation denotes the orientation of the curve.

(2) A parametric representation need not be unique.

(3) Note that using parameterisation reduces the number of variables at least by one.

5.6 Tangents and Normals to Conics

Tangent to a plane curve is a straight line touching the curve at exactly one point and a straight line perpendicular to the tangent and passing through the point of contact is called the normal at that point.

5.6.1 Equation of tangent and normal to the parabola $y^2 = 4ax$

(i) Equation of tangent in cartesian form

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a parabola $y^2 = 4ax$.

Then, $\quad y_1^2 = 4ax_1$ and $y_2^2 = 4ax_2$,

and $\quad y_1^2 - y_2^2 = 4a(x_1 - x_2)$.

Simplifying, $\quad \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{4a}{y_1 + y_2}$, the slope of the chord PQ.

Thus $\quad (y - y_1) = \dfrac{4a}{y_1 + y_2}(x - x_1)$, represents the equation of the chord PQ.

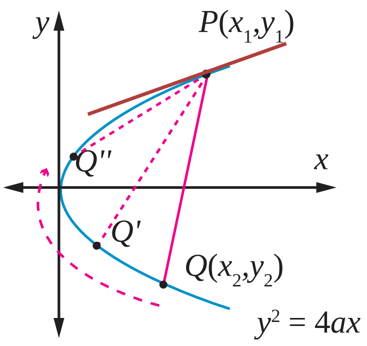


Fig. 5.48

When $Q \to P$, or $y_2 \to y_1$ the chord becomes tangent at P.

Thus the equation of tangent at (x_1, y_1) is

$$y - y_1 = \frac{4a}{2y_1}(x - x_1) \text{ where } \frac{2a}{y_1} \text{ is the slope of the tangent} \quad \ldots (1)$$

$$yy_1 - y_1^2 = 2ax - 2ax_1$$

$$yy_1 - 4ax_1 = 2ax - 2ax_1$$

$$\boxed{yy_1 = 2a(x + x_1)}$$

(ii) Equation of tangent in parametric form

Equation of tangent at $(at^2, 2at)$ on the parabola is

$$y(2at) = 2a(x + at^2)$$

$$\boxed{yt = x + at^2}$$

(iii) Equation of normal in cartesian form

From (1) the slope of normal is $-\dfrac{y_1}{2a}$

Therefore **equation of the normal** is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$$2ay - 2ay_1 = -y_1x + y_1x_1$$

$$xy_1 + 2ay = y_1(x_1 + 2a)$$

$$\boxed{xy_1 + 2ay = x_1y_1 + 2ay_1}$$

(iv) Equation of normal in parametric form

Equation of the normal at $(at^2, 2at)$ on the parabola is

$$x2at + 2ay = at^2(2at) + 2a(2at)$$

$$2a(xt + y) = 2a(at^3 + 2at)$$

$$\boxed{y + xt = at^3 + 2at}$$

Theorem 5.6

Three normals can be drawn to a parabola $y^2 = 4ax$ from a given point, one of which is always real.

Proof

$y^2 = 4ax$ is the given parabola. Let (α, β) be the given point.

Equation of the normal in parametric form is

$$y = -tx + 2at + at^3 \qquad \text{... (1)}$$

If m is the slope of the normal then $m = -t$.

Therefore the equation (1) becomes $\quad y = mx - 2am - am^3$.

Let it passes through (α, β), then

$$\beta = m\alpha - 2am - am^3$$

$$am^3 + (2a - \alpha)m + \beta = 0$$

which being a cubic equation in m, has three values of m. Consequently three normals, in general, can be drawn from a point to the parabola, since complex roots of real equation, always occur in conjugate pairs and (1) being an odd degree equation, it has atleast one real root. Hence atleast one normal to the parabola is real. ■

5.6.2 Equations of tangent and normal to Ellipse and Hyperbola (the proof of the following are left to the reader)

(1) Equation of the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(i) at (x_1, y_1) is $\dfrac{xx_1}{a^2} + \dfrac{yy_1}{b^2} = 1$ cartesian form

(ii) at $'\theta'$ $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$. parametric form

(2) Equation of the normal to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(i) at (x_1, y_1) is $\dfrac{a^2x}{x_1} - \dfrac{b^2y}{y_1} = a^2 - b^2$ cartesian form

(ii) at $'\theta'$ is $\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$ parametric form

(3) Equation of the tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i) at (x_1, y_1) is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$ cartesian form

(ii) at $'\theta'$ is $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$ parametric form

(4) Equation of the normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

(i) at (x_1, y_1) is $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$ cartesian form

(ii) at $'\theta'$ is $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$ parametric form.

5.6.3 Condition for the line $y = mx + c$ to be a tangent to the conic sections

(i) parabola $y^2 = 4ax$

Let (x_1, y_1) be the point on the parabola $y^2 = 4ax$. Then $y_1^2 = 4ax_1$ ... (1)

Let $y = mx + c$ be the tangent to the parabola ... (2)

Equation of tangent at (x_1, y_1) to the parabola from 5.6.1 is $yy_1 = 2a(x + x_1)$. ... (3)

Since (2) and (3) represent the same line, coefficients are proportional.

$$\frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{c}$$

$$\Rightarrow y_1 = \frac{2a}{m}, x_1 = \frac{c}{m}$$

Then (1) becomes, $$\left(\frac{2a}{m}\right)^2 = 4a\left(\frac{c}{m}\right)$$

$$\Rightarrow \boxed{c = \frac{a}{m}}$$

So the point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ and the equation of tangent to parabola is $y = mx + \frac{a}{m}$.

The condition for the line $y = mx + c$ to be tangent to the ellipse or hyperbola can be derived as follows in the same way as in the case of parabola.

(ii) ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Condition for line $y = mx + c$ to be the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$, with the point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$ and the equation of tangent is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

(iii) Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Condition for line $y = mx + c$ to be the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$, with the point of contact is $\left(-\frac{a^2m}{c}, -\frac{b^2}{c}\right)$ and the equation of tangent is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Note

(1) In $y = mx \pm \sqrt{a^2m^2 + b^2}$, either $y = mx + \sqrt{a^2m^2 + b^2}$ or $y = mx - \sqrt{a^2m^2 + b^2}$ is the equation to the tangent of ellipse but not both.

(2) In $y = mx \pm \sqrt{a^2m^2 - b^2}$, either $y = mx + \sqrt{a^2m^2 - b^2}$ or $y = mx - \sqrt{a^2m^2 - b^2}$ is the equation to the tangent of hyperbola but not both.

Results (Proof, left to the reader)

(1) Two tangents can be drawn to (i) a parabola (ii) an ellipse and (iii) a hyperbola, from any external point on the plane.

(2) Four normals can be drawn to (i) an ellipse and (ii) a hyperbola from any external point on the plane.

(3) The locus of the point of intersection of perpendicular tangents to

(i) the parabola $y^2 = 4ax$ is $x = -a$ (the directrix).

(ii) the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$ (called the director circle of ellipse).

(iii) the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$ (called director circle of hyperbola).

Example 5.29

Find the equations of tangent and normal to the parabola $x^2 + 6x + 4y + 5 = 0$ at $(1, -3)$.

Solution

Equation of parabola is $x^2 + 6x + 4y + 5 = 0$.

$$x^2 + 6x + 9 - 9 + 4y + 5 = 0$$

$$(x+3)^2 = -4(y-1) \qquad \dots (1)$$

$$\text{Let } X = x+3, Y = y-1$$

Equation (1) takes the standard form

$$X^2 = -4Y$$

Equation of tangent is $\quad XX_1 = -2(Y + Y_1)$

At $(1, -3)$ $\quad X_1 = 1 + 3 = 4; Y_1 = -3 - 1 = -4$

Therefore, the equation of tangent at $(1, -3)$ is

$$(x+3)4 = -2(y-1-4)$$

$$2x + 6 = -y + 5 .$$

$$2x + y + 1 = 0 .$$

Slope of tangent at $(1, -3)$ is -2, so slope of normal at $(1, -3)$ is $\frac{1}{2}$

Therefore, the equation of normal at $(1, -3)$ is given by

$$y + 3 = \frac{1}{2}(x-1)$$

$$2y + 6 = x - 1$$

$$x - 2y - 7 = 0 .$$

Example 5.30

Find the equations of tangent and normal to the ellipse $x^2 + 4y^2 = 32$ when $\theta = \frac{\pi}{4}$.

Solution

Equation of ellipse is
$$x^2+4y^2 = 32$$
$$\frac{x^2}{32}+\frac{y^2}{8} = 1$$
$$a^2 = 32,\ b^2=8$$
$$a = 4\sqrt{2},\ b=2\sqrt{2}$$

Equation of tangent at $\theta=\frac{\pi}{4}$ is
$$\frac{x\cos\frac{\pi}{4}}{4\sqrt{2}}+\frac{y\sin\frac{\pi}{4}}{2\sqrt{2}} = 1$$
$$\frac{x}{8}+\frac{y}{4} = 1$$
$$x+2y-8 = 0.$$

Equation of normal is
$$\frac{4\sqrt{2}x}{\cos\frac{\pi}{4}}-\frac{2\sqrt{2}y}{\sin\frac{\pi}{4}} = 32-8$$

That is
$$8x-4y = 24$$
$$2x-y-6 = 0.$$

Aliter

$$\text{At, } \theta = \frac{\pi}{4},$$
$$(a\cos\theta,\ b\sin\theta) = \left(4\sqrt{2}\cos\frac{\pi}{4},\ 2\sqrt{2}\sin\frac{\pi}{4}\right)$$
$$= (4,2)$$

$\therefore$ Equation of tangent at $\theta=\frac{\pi}{4}$ is same at $(4,2)$.

Equation of tangent in cartesian form is
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$$
$$x+2y-8 = 0$$

Slope of tangent is $-\frac{1}{2}$

Slope of normal is 2

Equation of normal is
$$y-2 = 2(x-4)$$
$$y-2x+6 = 0.$$

EXERCISE 5.4

1. Find the equations of the two tangents that can be drawn from $(5,2)$ to the ellipse $2x^2+7y^2=14$.
2. Find the equations of tangents to the hyperbola $\frac{x^2}{16}-\frac{y^2}{64}=1$ which are parallel to $10x-3y+9=0$.
3. Show that the line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$. Also find the coordinates of the point of contact.

4. Find the equation of the tangent to the parabola $y^2 = 16x$ perpendicular to $2x + 2y + 3 = 0$.

5. Find the equation of the tangent at $t = 2$ to the parabola $y^2 = 8x$. (Hint: use parametric form)

6. Find the equations of the tangent and normal to hyperbola $12x^2 - 9y^2 = 108$ at $\theta = \frac{\pi}{3}$. (Hint: use parametric form)

7. Prove that the point of intersection of the tangents at 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $\left[at_1t_2,\ a(t_1 + t_2)\right]$.

8. If the normal at the point 't_1' on the parabola $y^2 = 4ax$ meets the parabola again at the point 't_2', then prove that $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

5.7 Real life Applications of Conics

5.7.1 Parabola

The interesting applications of Parabola involve their use as reflectors and receivers of light or radio waves. For instance, cross sections of car headlights, flashlights are parabolas wherein the gadgets are formed by the paraboloid of revolution about its axis. The bulb in the headlights, flash lights is located at the focus and light from that point is reflected outward parallel to the axis of symmetry (Fig. 5.60) while Satellite dishes and field microphones used at sporting events, incoming radio waves or sound waves parallel to the axis that are reflected into the focus intensifying the same (Fig. 5.59). Similarly, in solar cooking, a parabolic mirror is mounted on a rack with a cooking pot hung in the focal area (Fig. 5.1). Incoming Sun rays parallel to the axis are reflected into the focus producing a temperature high enough for cooking.

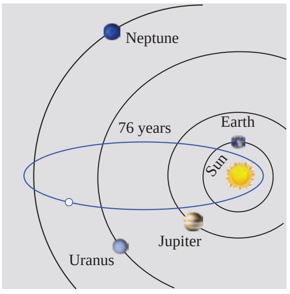
Parabolic arches are the best stable structures also considered for their beauty to name a few, the arches on the bridge of river in Godavari, Andhra Pradesh, India, the Eiffel tower in Paris, France.

Fig. 5.49

Fig. 5.50

5.7.2 Ellipse

According to Johannes Kepler, all planets in the solar system revolve around Sun in elliptic orbits with Sun at one of the foci. Some comets have elliptic orbits with Sun at one of the foci as well. E.g. Halley's Comet that is visible once every 75 years with $e \approx 0.97$ in elliptic orbit (Fig. 5.51). Our satellite moon travels around the Earth in an elliptical orbit with earth at one of its foci. Satellites of other planets also revolve around their planets in elliptical orbits as well.



The elliptical orbit of Halley's Comet

Fig. 5.51

Elliptic arches are often built for its beauty and stability. Steam boilers are believed to have greatest strength when heads are made elliptical with major and minor axes in the ratio 2:1.

In Bohr-Sommerfeld theory of the atom electron orbit can be circular or elliptical. Gears are sometimes (for particular need) made elliptical in shape. (Fig. 5.52)

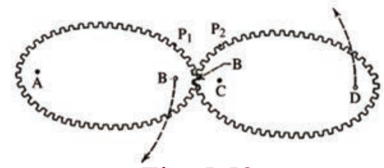


Fig. 5.52

The shape of our mother Earth is an oblate spheroid i.e., the solid of revolution of an ellipse about its minor axis, bulged along equatorial region and flat along the polar region.

The property of ellipse, any ray of light or sound released from a focus of the ellipse on touching the ellipse gets reflected to reach the other focus (Fig. 5.62), which could be proved using concepts of **incident rays** and **reflected rays** in Physics.

An exciting medical application of an ellipsoidal reflectors is a device called a **Lithotripter** (Fig. 5.4 and 5.63) that uses electromagnetic technology or ultrasound to generate a shock wave to pulverize kidney stones. The wave originates at one focus of the cross-sectional ellipse and is reflected to the kidney stone, which is positioned at the other focus. Recovery time following the use of this technique is much shorter than the conventional surgery, non-invasive and the mortality rate is lower.

5.7.3 Hyperbola

Some Comets travel in hyperbolic paths with the Sun at one focus, such comets pass by the Sun only one time unlike those in elliptical orbits, which reappear at intervals.

We also see hyperbolas in architecture, such as Mumbai Airport terminal (Fig. 5.53), in cross section of a planetarium, an locating ships (Fig. 5.54), or a cooling tower for a steam or nuclear power plant. (Fig. 5.5)


Fig. 5.53

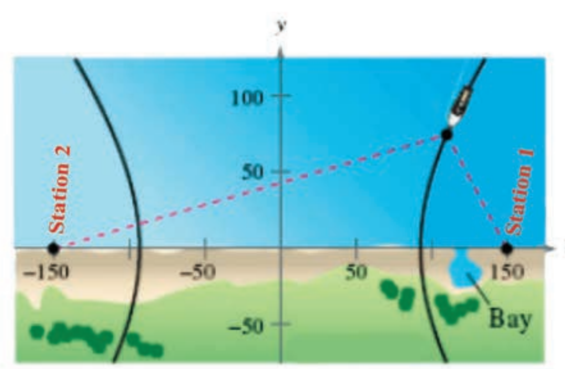


Fig. 5.54

Example 5.31

A semielliptical archway over a one-way road has a height of $3m$ and a width of $12m$. The truck has a width of $3m$ and a height of $2.7m$. Will the truck clear the opening of the archway? (Fig. 5.6)

Solution

Since the truck's width is $3m$, to determine the clearance, we must find the height of the archway $1.5m$ from the centre. If this height is $2.7m$ or less the truck will not clear the archway.

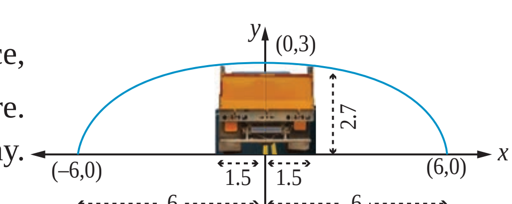


Fig. 5.55

From the diagram $a = 6$ and $b = 3$ yielding the equation of ellipse as $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$.

The edge of the $3m$ wide truck corresponds to $x = 1.5m$ from centre We will find the height of the archway $1.5m$ from the centre by substituting $x = 1.5$ and solving for y

$$\frac{\left(\frac{3}{2}\right)^2}{36} + \frac{y^2}{9} = 1$$

$$y^2 = 9\left(1 - \frac{9}{144}\right)$$

$$= \frac{9(135)}{144} = \frac{135}{16}$$

$$y = \frac{\sqrt{135}}{4}$$

$$= \frac{11.62}{4}$$

$$= 2.90$$

Thus the height of arch way $1.5m$ from the centre is approximately $2.90m$. Since the truck's height is $2.7m$, the truck will clear the archway. ■

Example 5.32

The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6 km. The Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

Solution

$$AS = 94.5 \times 10^6 \text{ km}, \; SA' = 152 \times 10^6 \text{ km}$$

$$a + c = 152 \times 10^6$$

$$a - c = 94.5 \times 10^6$$

Subtracting $2c = 57.5 \times 10^6 = 575 \times 10^5$ km

Distance of the Sun from the other focus is $SS' = 575 \times 10^5$ km. ■

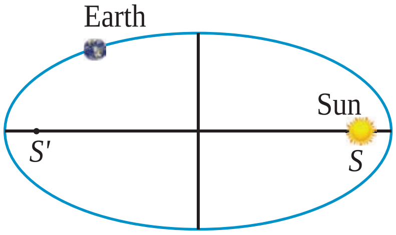


Fig. 5.56

Example 5.33

A concrete bridge is designed as a parabolic arch. The road over bridge is $40m$ long and the maximum height of the arch is $15m$. Write the equation of the parabolic arch.

Solution

From the graph the vertex is at $(0,0)$ and the parabola is open down

Equation of the parabola is $x^2 = -4ay$

$(-20,-15)$ and $(20,-15)$ lie on the parabola

$$20^2 = -4a(-15)$$

$$4a = \frac{400}{15}$$

$$x^2 = \frac{-80}{3} \times y$$

Therefore equation is $3x^2 = -80y$ ■

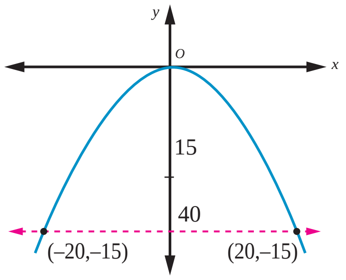


Fig. 5.57

Example 5.34

The parabolic communication antenna has a focus at $2m$ distance from the vertex of the antenna. Find the width of the antenna $3m$ from the vertex.

Solution

Let the parabola be $y^2 = 4ax$.

Since focus is $2m$ from the vertex $a = 2$

Equation of the parabola is $y^2 = 8x$

Let P be a point on the parabola whose x-coordinate is $3m$ from the vertex $P(3, y)$

$$y^2 = 8\times 3$$

$$y = \sqrt{8\times 3}$$

$$= 2\sqrt{6}$$

The width of the antenna $3m$ from the vertex is $4\sqrt{6}\,m$.

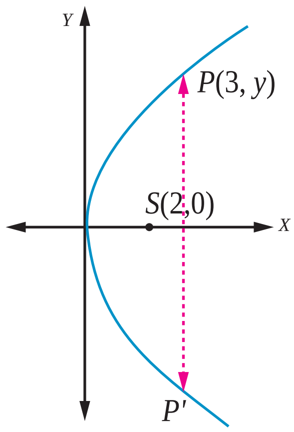


Fig. 5.58

5.7.4 Reflective property of parabola

The light or sound or radio waves originating at a parabola's focus are reflected parallel to the parabola's axis (Fig. 5.60) and conversely the rays arriving parallel to the axis are directed towards the focus (Fig. 5.59).

Example 5.35

The equation $y = \frac{1}{32}x^2$ models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of the parabola?

Solution

Equation of the parabola is

$$y = \frac{1}{32}x^2$$

That is $\quad x^2 = 32y$; the vertex is $(0,0)$

$$= 4(8)y$$

$$\Rightarrow a = 8$$

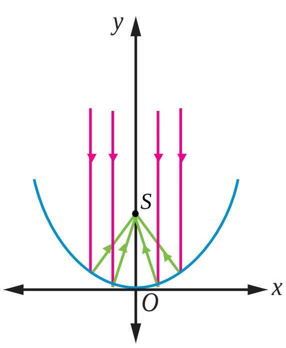


Fig. 5.59

So the heating tube needs to be placed at focus $(0, a)$. Hence the heating tube needs to be placed 8 units above the vertex of the parabola.

Example 5.36

A search light has a parabolic reflector (has a cross section that forms a 'bowl'). The parabolic bowl is $40\,cm$ wide from rim to rim and $30\,cm$ deep. The bulb is located at the focus .

(1) What is the equation of the parabola used for reflector?

(2) How far from the vertex is the bulb to be placed so that the maximum distance covered?

Solution

Let the vertex be $(0,0)$.

The equation of the parabola is

$$y^2 = 4ax$$

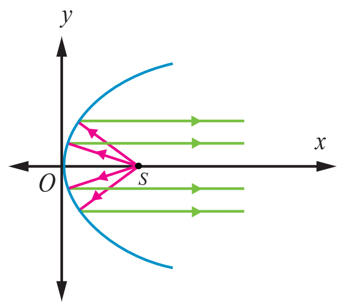


Fig. 5.60

(1) Since the diameter is $40\,cm$ and the depth is $30\,cm$, the point $(30, 20)$ lies on the parabola.

$$20^2 = 4a \times 30$$

$$4a = \frac{400}{30} = \frac{40}{3}.$$

$$\text{Equation is } y^2 = \frac{40}{3}x.$$

(2) The bulb is at focus $(a, 0)$. Hence the bulb is at a distance of $\frac{10}{3}cm$ from the vertex.

Example 5.37

An equation of the elliptical part of an optical lens system is $\frac{x^2}{16}+\frac{y^2}{9}=1$. The parabolic part of the system has a focus in common with the right focus of the ellipse .The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.

Solution

In the given ellipse $a^2 = 16,\ b^2 = 9$

then $\quad c^2 = a^2 - b^2$

$$c^2 = 16 - 9$$

$$= 7$$

$$c = \pm\sqrt{7}$$

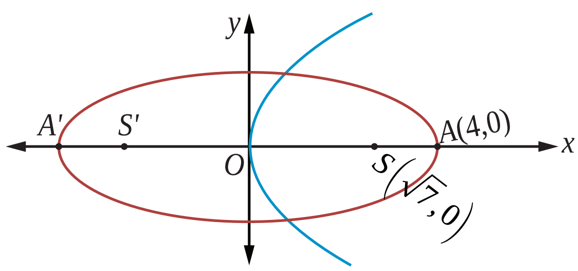


Fig. 5.61

Therefore the foci are $F\left(\sqrt{7},0\right), F'\left(-\sqrt{7},0\right)$. The focus of the parabola is $\left(\sqrt{7},0\right) \Rightarrow a = \sqrt{7}$.

Equation of the parabola is $y^2 = 4\sqrt{7}x$.

5.7.5 Reflective Property of an Ellipse

The lines from the foci to a point on an ellipse make equal angles with the tangent line at that point (Fig. 5.62).

The light or sound or radio waves emitted from one focus hits any point P on the ellipse is received at the other focus (Fig. 5.63).

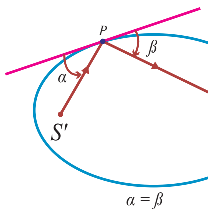


Fig. 5.62

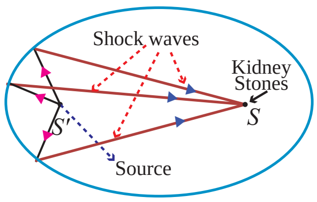


Fig. 5.63

Example 5.38

A room $34m$ long is constructed to be a whispering gallery. The room has an elliptical ceiling, as shown in Fig. 5.64. If the maximum height of the ceiling is $8m$, determine where the foci are located.

Solution

The length a of the semi major axis of the elliptical ceiling is $17m$. The height b of the semi minor axis is $8m$. Thus

$$c^2 = a^2 - b^2 = 17^2 - 8^2$$

$$\text{then} \quad c = \sqrt{289-64} = \sqrt{225}$$

$$\simeq 15$$

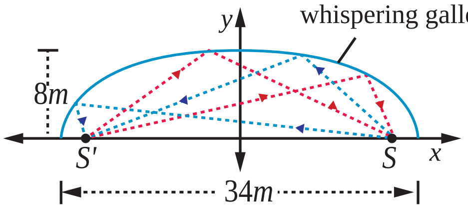


Fig. 5.64

For the elliptical ceiling the foci are located on either side about $15m$ from the centre, along its major axis.

A non-invasive medical miracle

In a lithotripter, a high-frequency sound wave is emitted from a source that is located at one of the foci of the ellipse. The patient is placed so that the kidney stone is located at the other focus of the ellipse.

Example 5.39

If the equation of the ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$ (x and y are measured in centimeters) where to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?

Solution

The equation of the ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$. The origin of the sound wave and the kidney stone of patient should be at the foci in order to crush the stones.

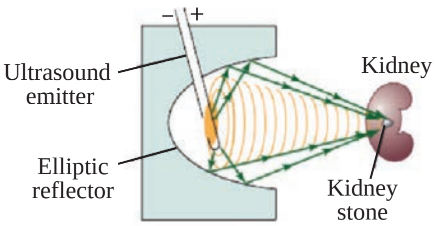


Fig. 5.65

$$a^2 = 484 \text{ and } b^2 = 64$$

$$c^2 = a^2 - b^2$$

$$= 484 - 64$$

$$= 420$$

$$c \simeq 20.5$$

Therefore the patient's kidney stone should be placed 20.5cm from the centre of the ellipse.

5.7.6 Reflective Property of a Hyperbola

The lines from the foci to a point on a hyperbola make equal angles with the tangent line at that point (Fig. 5.66).

The light or sound or radio waves directed from one focus is received at the other focus as in the case ellipse (Fig. 5.54) used in spotting location of ships sailing in deep sea.

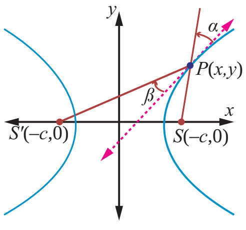


Fig. 5.66

Example 5.40

Two coast guard stations are located 600 km apart at points $A(0,0)$ and $B(0,600)$. A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200 km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

Solution

Since the centre is located at $(0,300)$, midway between the two foci, which are the coast guard stations, the equation is $\dfrac{(y-300)^2}{a^2}-\dfrac{(x-0)^2}{b^2}=1$. ... (1)

To determine the values of a and b, select two points known to be on the hyperbola and substitute each point in the above equation.

The point $(0,400)$ lies on the hyperbola, since it is 200 km further from Station A than from station B. $\dfrac{(400-300)^2}{a^2}-\dfrac{0}{b^2}=1$ $\dfrac{100^2}{a^2}=1$, $a^2=10000$. There is also a point $(x,600)$ on the hyperbola such that $600^2+x^2=(x+200)^2$.

$$360000+x^2 = x^2+400x+40000$$
$$x = 800$$

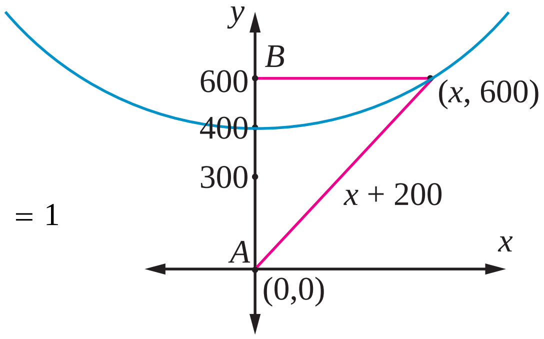


Fig. 5.67

Substituting in (1), we have $\dfrac{(600-300)^2}{10000}-\dfrac{(800-0)^2}{b^2}=1$

$$9-\frac{640000}{b^2} = 1$$
$$b^2 = 80000$$

Thus the required equation of the hyperbola is $\dfrac{(y-300)^2}{10000}-\dfrac{x^2}{80000}=1$

The ship lies somewhere on this hyperbola. The exact location can be determined using data from a third station.

Example 5.41

Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure 5.68 the parabola and hyperbola share focus F_1 which is $14m$ above the vertex of the parabola. The hyperbola's second focus F_2 is $2m$ above the parabola's vertex. The vertex of the hyperbolic mirror is $1m$ below F_1. Position a coordinate system with the origin at the centre of the hyperbola and with the foci on the y-axis. Then find the equation of the hyperbola.

Solution

Let V_1 be the vertex of the parabola and V_2 be the vertex of the hyperbola.

$$\overline{F_1F_2} = 14-2=12m,\ 2c=12,\ c=6$$

The distance of centre to the vertex of the hyperbola is $a=6-1=5$

$$b^2 = c^2-a^2$$
$$= 36-25=11.$$

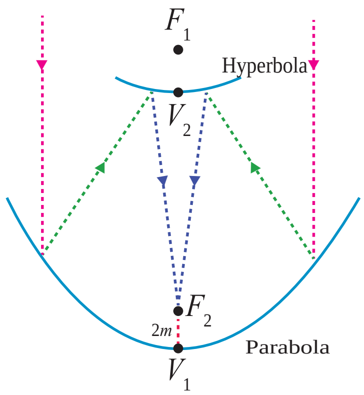


Fig. 5.68

Therefore the equation of the hyperbola is $\dfrac{y^2}{25}-\dfrac{x^2}{11}=1$.

EXERCISE 5.5

1. A bridge has a parabolic arch that is $10m$ high in the centre and $30m$ wide at the bottom. Find the height of the arch $6m$ from the centre, on either sides.

2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately . How wide must the opening be?

3. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5m$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75m$ from the point of origin.

4. An engineer designs a satellite dish with a parabolic cross section. The dish is $5m$ wide at the opening, and the focus is placed $1.2m$ from the vertex
   (a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.
   (b) Find the depth of the satellite dish at the vertex.

5. Parabolic cable of a $60m$ portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every $6m$ along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

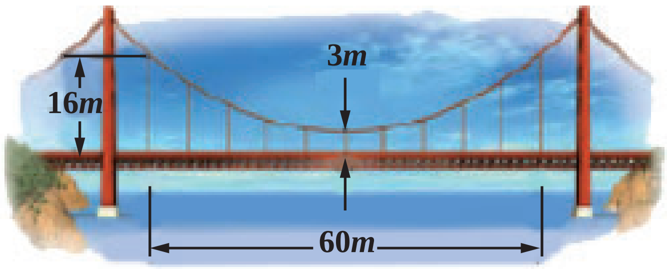


Fig. 5.69

6. Cross section of a Nuclear cooling tower is in the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is $150m$ tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.

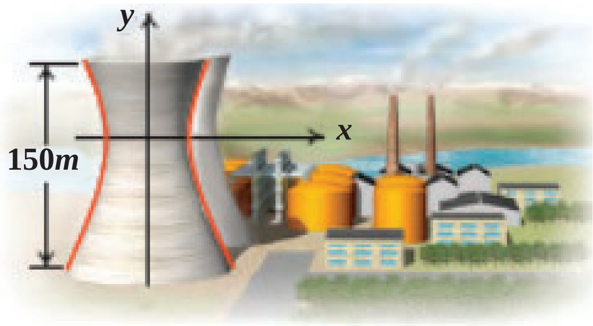


Fig. 5.70

7. A rod of length $1.2m$ moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is $0.3m$ from the end in contact with x-axis is an ellipse. Find the eccentricity.

8. Assume that water issuing from the end of a horizontal pipe, $7.5m$ above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position $2.5\,m$ below the line of the pipe, the flow of water has curved outward $3m$ beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of $4m$ when it is $6m$ away from the point of projection. Finally it reaches the ground $12m$ away from the starting point. Find the angle of projection.

10. Points A and B are $10km$ apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is $6\ km$ closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

EXERCISE 5.6

6DPUC

Choose the correct or the most suitable answer from the given four alternatives :

1. The equation of the circle passing through $(1,5)$ and $(4,1)$ and touching y-axis is $x^2+y^2-5x-6y+9+\lambda(4x+3y-19)=0$ where λ is equal to

   (1) $0,-\dfrac{40}{9}$ (2) 0 (3) $\dfrac{40}{9}$ (4) $\dfrac{-40}{9}$

2. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is

   (1) $\dfrac{4}{3}$ (2) $\dfrac{4}{\sqrt{3}}$ (3) $\dfrac{2}{\sqrt{3}}$ (4) $\dfrac{3}{2}$

3. The circle $x^2+y^2=4x+8y+5$ intersects the line $3x-4y=m$ at two distinct points if

   (1) $15<m<65$ (2) $35<m<85$ (3) $-85<m<-35$ (4) $-35<m<15$

4. The length of the diameter of the circle which touches the x-axis at the point $(1,0)$ and passes through the point $(2,3)$.

   (1) $\dfrac{6}{5}$ (2) $\dfrac{5}{3}$ (3) $\dfrac{10}{3}$ (4) $\dfrac{3}{5}$

5. The radius of the circle $3x^2+by^2+4bx-6by+b^2=0$ is

   (1) 1 (2) 3 (3) $\sqrt{10}$ (4) $\sqrt{11}$

6. The centre of the circle inscribed in a square formed by the lines $x^2-8x-12=0$ and $y^2-14y+45=0$ is

   (1) $(4,7)$ (2) $(7,4)$ (3) $(9,4)$ (4) $(4,9)$

7. The equation of the normal to the circle $x^2+y^2-2x-2y+1=0$ which is parallel to the line $2x+4y=3$ is

(1) $x+2y=3$ (2) $x+2y+3=0$ (3) $2x+4y+3=0$ (4) $x-2y+3=0$

8. If $P(x,y)$ be any point on $16x^2+25y^2=400$ with foci $F_1(3,0)$ and $F_2(-3,0)$ then PF_1+PF_2 is

(1) 8 (2) 6 (3) 10 (4) 12

9. The radius of the circle passing through the point $(6,2)$ two of whose diameter are $x+y=6$ and $x+2y=4$ is

(1) 10 (2) $2\sqrt{5}$ (3) 6 (4) 4

10. The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is

(1) $4(a^2+b^2)$ (2) $2(a^2+b^2)$ (3) a^2+b^2 (4) $\frac{1}{2}(a^2+b^2)$

11. If the normals of the parabola $y^2=4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2+(y+2)^2=r^2$, then the value of r^2 is

(1) 2 (2) 3 (3) 1 (4) 4

12. If $x+y=k$ is a normal to the parabola $y^2=12x$, then the value of k is

(1) 3 (2) -1 (3) 1 (4) 9

13. The ellipse $E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0,4)$ circumscribes the rectangle R. The eccentricity of the ellipse is

(1) $\frac{\sqrt{2}}{2}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{3}{4}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ parallel to the straight line $2x-y=1$. One of the points of contact of tangents on the hyperbola is

(1) $\left(\frac{9}{2\sqrt{2}},\frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (3) $\left(\frac{9}{2\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ (4) $\left(3\sqrt{3},-2\sqrt{2}\right)$

15. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ having centre at $(0,3)$ is

(1) $x^2+y^2-6y-7=0$ (2) $x^2+y^2-6y+7=0$

(3) $x^2+y^2-6y-5=0$ (4) $x^2+y^2-6y+5=0$

16. Let C be the circle with centre at $(1,1)$ and radius $=1$. If T is the circle centered at $(0, y)$ passing through the origin and touching the circle C externally, then the radius of T is equal to

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

17. Consider an ellipse whose centre is of the origin and its major axis is along x-axis. If its eccentrcity is $\frac{3}{5}$ and the distance between its foci is 6, then the area of the quadrilateral inscribed in the ellipse with diagonals as major and minor axis of the ellipse is

(1) 8 (2) 32 (3) 80 (4) 40

18. Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(1) $2ab$ (2) ab (3) $\sqrt{ab}$ (4) $\frac{a}{b}$

19. An ellipse has OB as semi minor axes, F and F' its foci and the angle FBF' is a right angle. Then the eccentricity of the ellipse is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{3}}$

20. The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{3}$ (3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}$

21. If the two tangents drawn from a point P to the parabola $y^2=4x$ are at right angles then the locus of P is

(1) $2x+1=0$ (2) $x=-1$ (3) $2x-1=0$ (4) $x=1$

22. The circle passing through $(1,-2)$ and touching the axis of x at $(3,0)$ passing through the point

(1) $(-5,2)$ (2) $(2,-5)$ (3) $(5,-2)$ (4) $(-2,5)$

23. The locus of a point whose distance from $(-2,0)$ is $\frac{2}{3}$ times its distance from the line $x=\frac{-9}{2}$ is

(1) a parabola (2) a hyperbola (3) an ellipse (4) a circle

24. The values of m for which the line $y=mx+2\sqrt{5}$ touches the hyperbola $16x^2-9y^2=144$ are the roots of $x^2-(a+b)x-4=0$, then the value of $(a+b)$ is

(1) 2 (2) 4 (3) 0 (4) -2

25. If the coordinates at one end of a diameter of the circle $x^2+y^2-8x-4y+c=0$ are $(11,2)$, the coordinates of the other end are

(1) $(-5,2)$ (2) $(-3, 2)$ (3) $(5,-2)$ (4) $(-2,5)$

SUMMARY

(1) Equation of the circle in a standard form is $(x-h)^2+(y-k)^2=r^2$.

(i) Centre (h,k) (ii) radius 'r'

(2) Equation of a circle in general form is $x^2+y^2+2gx+2fy+c=0$.

(i) centre $(-g,-f)$ (ii) radius = $\sqrt{g^2+f^2-c}$

(3) The circle through the intersection of the line $lx+my+n=0$ and the circle

$x^2+y^2+2gx+2fy+c=0$ is $x^2+y^2+2gx+2fy+c+\lambda(lx+my+n)=0, \lambda\in\mathbb{R}^1$.

(4) Equation of a circle with (x_1,y_1) and (x_2,y_2) as extremities of one of the diameters is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

(5) Equation of tangent at (x_1,y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

(6) Equation of normal at (x_1,y_1) on circle $x^2+y^2+2gx+2fy+c=0$ is

$yx_1-xy_1+g(y-y_1)-f(x-x_1)=0$.

Table 1

Tangent and normal

Curve	Equation	Equation of tangent	Equation of normal
Circle	$x^2+y^2=a^2$	(i) cartesian form $xx_1+yy_1=a^2$ (ii) parametric form $x\cos\theta+y\sin\theta=a$	(i) cartesian form $xy_1-yx_1=0$ (ii) parametric form $x\sin\theta-y\cos\theta=0$
Parabola	$y^2=4ax$	(i) $yy_1=2a(x+x_1)$ (ii) $yt=x+at^2$	(i) $xy_1+2y=2ay_1+x_1y_1$ (ii) $y+xt=at^3+2at$
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	(i) $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ (ii) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$	(i) $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2-b^2$ (ii) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	(i) $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$ (ii) $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$	(i) $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$ (ii) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

Table 2

Condition for the sine $y = mx + c$ to be a tangent to the Conics

Conic	Equation	Condition to be tangent	Point of contact	Equation of tangent
Circle	$x^2 + y^2 = a^2$	$c^2 = a^2(1+m^2)$	$\left(\frac{\mp am}{\sqrt{1+m^2}}, \frac{\pm a}{\sqrt{1+m^2}}\right)$	$y = mx \pm \sqrt{1+m^2}$
Parabola	$y^2 = 4ax$	$c = \frac{a}{m}$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$y = mx + \frac{a}{m}$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 + b^2$	$\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 + b^2}$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$c^2 = a^2m^2 - b^2$	$\left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$	$y = mx \pm \sqrt{a^2m^2 - b^2}$

Table 3

Parametric forms

Conic	Parametric equations	Parameter	Range of parameter	Any point on the conic
Circle	$x = a\cos\theta$ $y = a\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, a\sin\theta)$
Parabola	$x = at^2$ $y = 2at$	t	$-\infty < t < \infty$	't' or $(at^2, 2at)$
Ellipse	$x = a\cos\theta$ $y = b\sin\theta$	θ	$0 \le \theta \le 2\pi$	'θ' or $(a\cos\theta, b\sin\theta)$
Hyperbola	$x = a\sec\theta$ $y = b\tan\theta$	θ	$-\pi \le \theta \le \pi$ except $\theta = \pm\frac{\pi}{2}$	'θ' or $(a\sec\theta, b\tan\theta)$

Identifying the conic from the general equation of conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

The graph of the second degree equation is one of a circle, parabola, an ellipse, a hyperbola, a point, an empty set, a single line or a pair of lines. When,

(1) $A = C = 1,\ B = 0, D = -2h,\ E = -2k,\ F = h^2 + k^2 - r^2$ the general equation reduces to $(x-h)^2 + (y-k)^2 = r^2$, which is a circle.

(2) $B=0$ and either A or $C=0$, the general equation yields a parabola under study, at this level.

(3) $A \neq C$ and A and C are of the same sign the general equation yields an ellipse.

(4) $A \neq C$ and A and C are of opposite signs the general equation yields a hyperbola

(5) $A=C$ and $B=D=E=F=0$, the general equation yields a point $x^2+y^2=0$.

(6) $A=C=F$ and $B=D=E=0$, the general equation yields an empty set $x^2+y^2+1=0$, as there is no real solution.

(7) $A \neq 0$ or $C \neq 0$ and others are zeros, the general equation yield coordinate axes.

(8) $A=-C$ and rests are zero, the general equation yields a pair of lines $x^2-y^2=0$.

ICT CORNER

https://ggbm.at/vchq92pg or Scan the QR Code

Open the Browser, type the URL Link given below (or) Scan the QR code. GeoGebra work book named "12th Standard Mathematics" will open. In the left side of the work book there are many chapters related to your text book. Click on the chapter named "Two Dimensional Analytical Geometry-II". You can see several work sheets related to the chapter. Select the work sheet "Conic Tracing"

Chapter 6

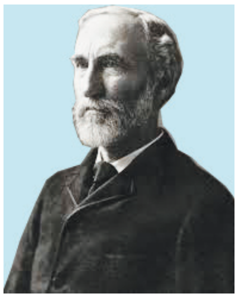
Applications of Vector Algebra

BD9XA

"Mathematics is the science of the connection of magnitudes.
Magnitude is anything that can be put equal or unequal to another thing.
Two things are equal when in every assertion each may be replaced by the other."

— Hermann Günther Grassmann

6.1 Introduction



Josiah Williard Gibbs
(1839 – 1903)

We are familiar with the concept of vectors, (***vectus*** in Latin means "to carry") from our XI standard text book. Further the modern version of Theory of Vectors arises from the ideas of Wessel(1745-1818) and Argand (1768-1822) when they attempt to describe the complex numbers geometrically as a directed line segment in a coordinate plane. We have seen that **a vector has magnitude and direction** and **two vectors with same magnitude and direction regardless of positions of their initial points are always equal**.

We also have studied addition of two vectors, scalar multiplication of vectors, dot product, and cross product by denoting an arbitrary vector by the notation $\vec{a}$ or $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$. To understand the direction and magnitude of a given vector and all other concepts with a little more rigor, we shall recall the geometric introduction of vectors, which will be useful to discuss the equations of straight lines and planes. Great mathematicians Grassmann, Hamilton, Clifford and Gibbs were pioneers to introduce the dot and cross products of vectors.

The vector algebra has a few direct applications in physics and it has a lot of applications along with vector calculus in physics, engineering, and medicine. Some of them are mentioned below.

- To calculate the volume of a parallelepiped, the scalar triple product is used.
- To find the work done and torque in mechanics, the dot and cross products are respectiveluy used.
- To introduce curl and divergence of vectors, vector algebra is used along with calculus. Curl and divergence are very much used in the study of electromagnetism, hydrodynamics, blood flow, rocket launching, and the path of a satellite.
- To calculate the distance between two aircrafts in the space and the angle between their paths, the dot and cross products are used.
- To install the solar panels by carefully considering the tilt of the roof, and the direction of the Sun so that it generates more solar power, a simple application of dot product of vectors is used. One can calculate the amount of solar power generated by a solar panel by using vector algebra.
- To measure angles and distance between the panels in the satellites, in the construction of networks of pipes in various industries, and, in calculating angles and distance between beams and structures in civil engineering, vector algebra is used.

Learning Objectives

Upon completion of this chapter, students will be able to

- apply scalar and vector products of two and three vectors
- solve problems in geometry, trigonometry, and physics
- derive equations of a line in parametric, non-parametric, and cartesian forms in different situations
- derive equations of a plane in parametric, non-parametric, and cartesian forms in different situations
- find angle between the lines and distance between skew lines
- find the coordinates of the image of a point

6.2 Geometric introduction to vectors

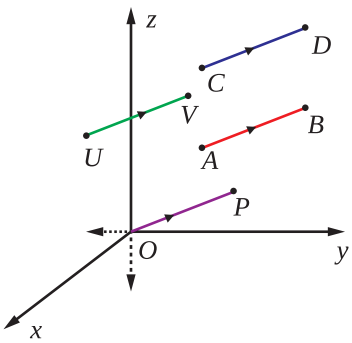


Fig. 6.1

A vector $\vec{v}$ is represented as a directed straight line segment in a 3-dimensional space $\mathbb{R}^3$, with an initial point $A = (a_1, a_2, a_3) \in \mathbb{R}^3$ and an end point $B = (b_1, b_2, b_3) \in \mathbb{R}^3$, and it is denoted by $\overrightarrow{AB}$. The length of the line segment AB is the magnitude of the vector $\vec{v}$ and the direction from A to B is the direction of the vector $\vec{v}$. Hereafter, a vector will be interchangeably denoted by $\vec{v}$ or $\overrightarrow{AB}$. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ in $\mathbb{R}^3$ are said to be equal if and only if the length AB is equal to the length CD and the direction from A to B is parallel to the direction from C to D. If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal, we write $\overrightarrow{AB} = \overrightarrow{CD}$, and $\overrightarrow{CD}$ is called a **translate** of $\overrightarrow{AB}$.

It is easy to observe that every vector $\overrightarrow{AB}$ can be translated to anywhere in $\mathbb{R}^3$, equal to a vector with initial point $U \in \mathbb{R}^3$ and end point $V \in \mathbb{R}^3$ such that $\overrightarrow{AB} = \overrightarrow{UV}$. In particular, if O is the origin of $\mathbb{R}^3$, then a point $P \in \mathbb{R}^3$ can be found such that $\overrightarrow{AB} = \overrightarrow{OP}$. The vector $\overrightarrow{OP}$ is called the **position vector** of the point P. Moreover, we observe that given any vector $\vec{v}$, there exists a unique point $P \in \mathbb{R}^3$ such that the position vector $\overrightarrow{OP}$ of P is equal to $\vec{v}$. A vector $\overrightarrow{AB}$ is said to be the **zero vector** if the initial point A is the same as the end point B. We use the standard notations $\hat{i}, \hat{j}, \hat{k}$ and $\vec{0}$ to denote the position vectors of the points $(1,0,0), (0,1,0), (0,0,1)$, and $(0,0,0)$, respectively. For a given point $(a_1, a_2, a_3) \in \mathbb{R}^3$, $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is called the position vector of the point (a_1, a_2, a_3), which is the directed straight line segment with initial point $(0,0,0)$ and end point (a_1, a_2, a_3). All real numbers are called scalars.

Given a vector $\overrightarrow{AB}$, the **length of the vector** is calculated by

$$\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}\,,$$

where A is (a_1,a_2,a_3) and B is (b_1,b_2,b_3). In particular, if a vector is the position vector $\vec{b}$ of (b_1,b_2,b_3), then its length is $\sqrt{b_1{}^2+b_2{}^2+b_3{}^2}$. A vector having length 1 is called a **unit vector**. We use the notation $\hat{u}$, for a unit vector. Note that $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors and $\vec{0}$ is the unique vector with length 0. The direction of $\vec{0}$ is specified according to the context.

The **addition** and **scalar multiplication** on vectors in 3-dimensional space are defined by

$$\vec{a}+\vec{b} \;=\; (a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}\,.$$

$$\alpha\vec{a} \;=\; (\alpha a_1)\hat{i}+(\alpha a_2)\hat{j}+(\alpha a_3)\hat{k}\,;$$

where $\quad \vec{a} \;=\; a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\;\; \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\in\mathbb{R}^3$ and $\alpha\in\mathbb{R}$.

To see the geometric interpretation of $\vec{a}+\vec{b}$, let $\vec{a}$ and $\vec{b}$, denote the position vectors of $A=(a_1,a_2,a_3)$ and $B=(b_1,b_2,b_3)$, respectively. Translate the position vector $\vec{b}$ to the vector with initial point as A and end point as $C=(c_1,c_2,c_3)$, for a suitable $(c_1,c_2,c_3)\in\mathbb{R}^3$. See the Fig (6.2). Then, the position vector $\vec{c}$ of the point (c_1,c_2,c_3) is equal to $\vec{a}+\vec{b}$.

The vector $\alpha\vec{a}$ is another vector parallel to $\vec{a}$ and its length is magnified (if $\alpha>1$) or contracted (if $0<\alpha<1$). If $\alpha<0$, then $\alpha\vec{a}$ is a vector whose magnitude is $|\alpha|$ times that of $\vec{a}$ and direction opposite to that of $\vec{a}$. In particular, if $\alpha=-1$, then $\alpha\vec{a}=-\vec{a}$ is the vector with same length and direction opposite to that of $\vec{a}$. See Fig. 6.3

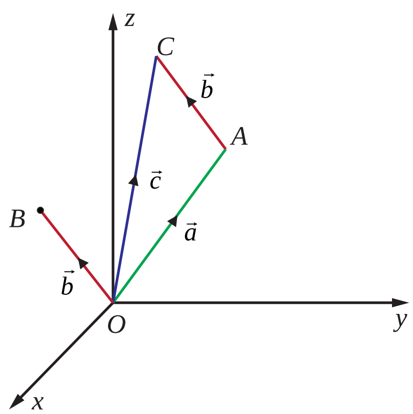


Fig. 6.2

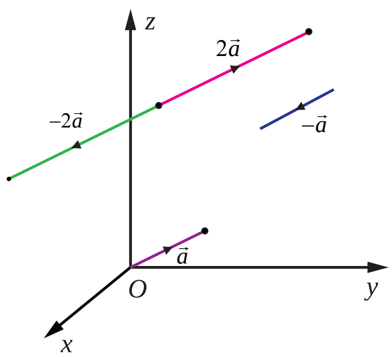


Fig. 6.3

6.3 Scalar Product and Vector Product

Next we recall the scalar product and vector product of two vectors as follows.

Definition 6.1

Given two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ the scalar product (or dot product) is denoted by $\vec{a}\cdot\vec{b}$ and is calculated by

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3,$$

and the vector product (or cross product) is denoted by $\vec{a}\times\vec{b}$, and is calculated by

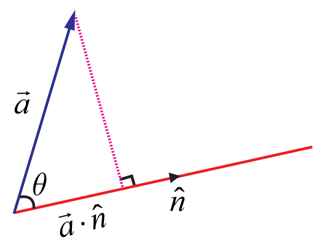
$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Note

$\vec{a}\cdot\vec{b}$ is a scalar, and $\vec{a}\times\vec{b}$ is a vector.

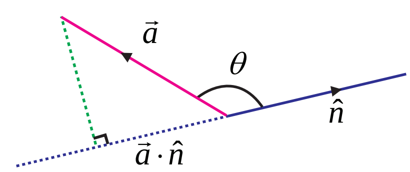
6.3.1 Geometrical interpretation

Geometrically, if $\vec{a}$ is an arbitrary vector and $\hat{n}$ is a unit vector, then $\vec{a}\cdot\hat{n}$ is the projection of the vector $\vec{a}$ on the straight line on which $\hat{n}$ lies. The quantity $\vec{a}\cdot\hat{n}$ is positive if the angle between $\vec{a}$ and $\hat{n}$ is acute, see Fig. 6.4 and negative if the angle between $\vec{a}$ and $\hat{n}$ is obtuse see Fig. 6.5.



Positive dot product

Fig. 6.4



Negative dot product

Fig. 6.5

If $\vec{a}$ and $\vec{b}$ are arbitrary non-zero vectors, then $|\vec{a}\cdot\vec{b}| = \left| |\vec{b}|\,\vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right)\right| = \left| |\vec{a}|\,\vec{b}\cdot\left(\frac{\vec{a}}{|\vec{a}|}\right)\right|$ and so $|\vec{a}\cdot\vec{b}|$ means either the length of the straight line segment obtained by projecting the vector $|\vec{b}|\vec{a}$ along the direction of $\vec{b}$ or the length of the line segment obtained by projecting the vector $|\vec{a}|\vec{b}$ along the direction of $\vec{a}$. We recall that $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$. We recall that the angle between $\vec{a}$ and $\vec{b}$ is defined as the measure from $\vec{a}$ to $\vec{b}$ in the counter clockwise direction.

The vector $\vec{a}\times\vec{b}$ is either $\vec{0}$ or a vector perpendicular to the plane parallel to both $\vec{a}$ and $\vec{b}$ having magnitude as the area of the parallelogram formed by coterminus vectors parallel to $\vec{a}$ and $\vec{b}$. If $\vec{a}$ and $\vec{b}$ are non-zero vectors, then the magnitude of $\vec{a}\times\vec{b}$ can be calculated by the formula

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta|, \text{ where } \theta \text{ is the angle between } \vec{a} \text{ and } \vec{b}.$$

Two vectors are said to be coterminus if they have same initial point.

Remark

(1) An angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is found by the following formula $\theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$.

(2) $\vec{a}$ and $\vec{b}$ are said to be parallel if the angle between them is 0 or π.

(3) $\vec{a}$ and $\vec{b}$ are said to be perpendicular if the angle between them is $\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$.

Property

(1) Let $\vec{a}$ and $\vec{b}$ be any two nonzero vectors. Then

- $\vec{a}\cdot\vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other.
- $\vec{a}\times\vec{b} = \vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel to each other.

(2) If $\vec{a},\vec{b}$, and $\vec{c}$ are any three vectors and α is a scalar, then

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a},\ (\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c},\ (\alpha\vec{a})\cdot\vec{b} = \alpha(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(\alpha\vec{b});$$

$$\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a}),\ (\vec{a}+\vec{b})\times\vec{c} = \vec{a}\times\vec{c}+\vec{b}\times\vec{c},\ (\alpha\vec{a})\times\vec{b} = \alpha(\vec{a}\times\vec{b}) = \vec{a}\times(\alpha\vec{b}).$$

6.3.2 Application of dot and cross products in plane Trigonometry

We apply the concepts of dot and cross products of two vectors to derive a few formulae in plane trigonometry.

Example 6.1 (Cosine formulae)

With usual notations, in any triangle *ABC*, prove the following by vector method.

(i) $a^2 = b^2 + c^2 - 2bc\cos A$ (ii) $b^2 = c^2 + a^2 - 2ca\cos B$

(iii) $c^2 = a^2 + b^2 - 2ab\cos C$

Solution

With usual notations in triangle *ABC*, we have $\overrightarrow{BC} = \vec{a}, \overrightarrow{CA} = \vec{b}$ and $\overrightarrow{AB} = \vec{c}$. Then $|\overrightarrow{BC}| = a, |\overrightarrow{CA}| = b$, $|\overrightarrow{AB}| = c$ and $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB} = \vec{0}$.

So, $\overrightarrow{BC} = -\overrightarrow{CA}-\overrightarrow{AB}$.

Then applying dot product, we get

$$\overrightarrow{BC}\cdot\overrightarrow{BC} = (-\overrightarrow{CA}-\overrightarrow{AB})\cdot(-\overrightarrow{CA}-\overrightarrow{AB})$$

$$\Rightarrow \quad |\overrightarrow{BC}|^2 = |\overrightarrow{CA}|^2 + |\overrightarrow{AB}|^2 + 2\overrightarrow{CA}\cdot\overrightarrow{AB}$$

$$\Rightarrow \quad a^2 = b^2 + c^2 + 2bc\cos(\pi - A)$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A.$$

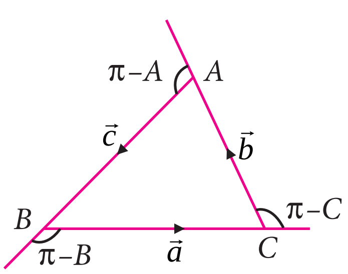


Fig. 6.6

The results in (ii) and (iii) are proved in a similar way. ■

Example 6.2

With usual notations, in any triangle *ABC*, prove the following by vector method.

(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

Solution

With usual notations in triangle ABC, we have $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, and $\overrightarrow{AB}=\vec{c}$. Then

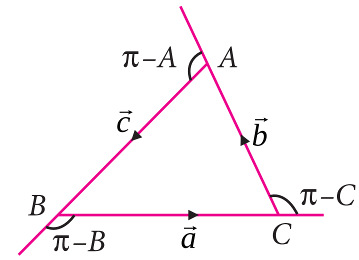


Fig. 6.7

$$|\overrightarrow{BC}|=a,\ |\overrightarrow{CA}|=b,\ |\overrightarrow{AB}|=c \text{ and } \overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=\vec{0}$$

So, $\overrightarrow{BC}=-\overrightarrow{CA}-\overrightarrow{AB}$

Applying dot product, we get

$$\overrightarrow{BC}\cdot\overrightarrow{BC} = -\overrightarrow{BC}\cdot\overrightarrow{CA}-\overrightarrow{BC}\cdot\overrightarrow{AB}$$

$$\Rightarrow\ |\overrightarrow{BC}|^2 = -|\overrightarrow{BC}|\,|\overrightarrow{CA}|\ \cos(\pi-C)-|\overrightarrow{BC}|\,|\overrightarrow{AB}|\cos(\pi-B)$$

$$\Rightarrow\quad a^2 = ab\cos C+ac\cos B$$

Therefore $a=b\cos C+c\cos B$. The results in (ii) and (iii) are proved in a similar way. ■

Example 6.3

By vector method, prove that $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

Solution

Let $\hat{a}=\overrightarrow{OA}$ and $\hat{b}=\overrightarrow{OB}$ be the unit vectors and which make angles α and β, respectively, with positive x-axis, where A and B are as in the Fig. 6.8. Draw AL and BM perpendicular to the x-axis. Then $|\overrightarrow{OL}|=|\overrightarrow{OA}|\cos\alpha=\cos\alpha$, $|\overrightarrow{LA}|=|\overrightarrow{OA}|\sin\alpha=\sin\alpha$.

So, $\overrightarrow{OL}=|\overrightarrow{OL}|\,\hat{i} = \cos\alpha\hat{i}$, $\overrightarrow{LA}=\sin\alpha(-\hat{j})$.

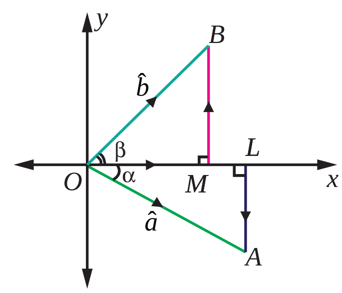


Fig. 6.8

Therefore, $\hat{a} = \overrightarrow{OA}=\overrightarrow{OL}+\overrightarrow{LA}=\cos\alpha\hat{i}-\sin\alpha\hat{j}$. ... (1)

Similarly, $\hat{b} = \cos\beta\hat{i}+\sin\beta\hat{j}$ ... (2)

The angle between $\hat{a}$ and $\hat{b}$ is $\alpha+\beta$ and so,

$$\hat{a}\cdot\hat{b} = |\hat{a}|\ |\hat{b}|\cos(\alpha+\beta)=\cos(\alpha+\beta) \quad \text{... (3)}$$

On the other hand, from (1) and (2)

$$\hat{a}\cdot\hat{b} = (\cos\alpha\hat{i}-\sin\alpha\hat{j})\cdot(\cos\beta\hat{i}+\sin\beta\hat{j})=\cos\alpha\cos\beta-\sin\alpha\sin\beta. \quad \text{... (4)}$$

From (3) and (4), we get $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$. ■

Example 6.4

With usual notations, in any triangle ABC, prove by vector method that $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

Solution

With usual notations in triangle ABC, we have $\overrightarrow{BC}=\vec{a}$, $\overrightarrow{CA}=\vec{b}$, and $\overrightarrow{AB}=\vec{c}$. Then $|\overrightarrow{BC}|=a$, $|\overrightarrow{CA}|=b$, and $|\overrightarrow{AB}|=c$.

Since in ΔABC, $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=0$, we have $\overrightarrow{BC}\times(\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB})=\vec{0}$.

Simplification gives,

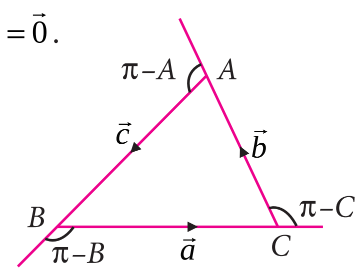


Fig. 6.9

$$\overrightarrow{BC}\times\overrightarrow{CA} = \overrightarrow{AB}\times\overrightarrow{BC}. \quad \text{... (1)}$$

Similarly, since $\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}=\vec{0}$, we have

$$\overrightarrow{CA}\times(\overrightarrow{BC}+\overrightarrow{CA}+\overrightarrow{AB}) = \vec{0}.$$

On Simplification, we obtain $\overrightarrow{BC}\times\overrightarrow{CA} = \overrightarrow{CA}\times\overrightarrow{AB}$ ... (2)

Equations (1) and (2), we get

$$\overrightarrow{AB}\times\overrightarrow{BC} = \overrightarrow{CA}\times\overrightarrow{AB} = \overrightarrow{BC}\times\overrightarrow{CA}.$$

So, $|\overrightarrow{AB}\times\overrightarrow{BC}| = |\overrightarrow{CA}\times\overrightarrow{AB}| = |\overrightarrow{BC}\times\overrightarrow{CA}|$. Then, we get

$$ca\sin(\pi - B) = bc\sin(\pi - A) = ab\sin(\pi - C).$$

That is, $ca\sin B = bc\sin A = ab\sin C$. Dividing by abc, leads to

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$ ■

Example 6.5

Prove by vector method that $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Solution

Let $\hat{a} = \overrightarrow{OA}$ and $\hat{b} = \overrightarrow{OB}$ be the unit vectors making angles α and β respectively, with positive x-axis, where A and B are as shown in the Fig. 6.10. Then, we get $\hat{a} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$ and $\hat{b} = \cos\beta\hat{i} + \sin\beta\hat{j}$,

The angle between $\hat{a}$ and $\hat{b}$ is $\alpha - \beta$ and, the vectors $\hat{b}, \hat{a}, \hat{k}$ form a right-handed system.

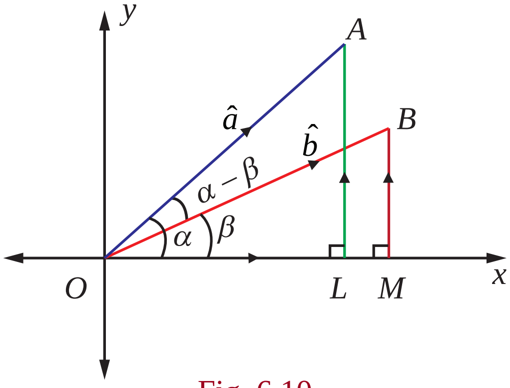


Fig. 6.10

Hence, we get

$$\hat{b}\times\hat{a} = |\hat{b}||\hat{a}|\sin(\alpha - \beta)\hat{k} = \sin(\alpha - \beta)\hat{k}. \quad \text{... (1)}$$

On the other hand,

$$\hat{b}\times\hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\hat{k} \quad \text{... (2)}$$

Hence, equations (1) and (2), leads to

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$ ■

6.3.3 Application of dot and cross products in Geometry

Example 6.6 (Apollonius's theorem)

If D is the midpoint of the side BC of a triangle ABC, show by vector method that $|\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 = 2(|\overrightarrow{AD}|^2 + |\overrightarrow{BD}|^2)$.

Solution

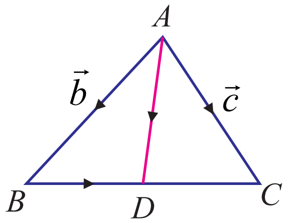


Fig. 6.11

Let A be the origin, $\vec{b}$ be the position vector of B and $\vec{c}$ be the position vector of C. Now D is the midpoint of BC, and so the, position vector of D is $\frac{\vec{b}+\vec{c}}{2}$. Therefore, we have

$$|\overrightarrow{AD}|^2 = \overrightarrow{AD}\cdot\overrightarrow{AD} = \left(\frac{\vec{b}+\vec{c}}{2}\right)\cdot\left(\frac{\vec{b}+\vec{c}}{2}\right) = \frac{1}{4}(|\vec{b}|^2+|\vec{c}|^2+2\vec{b}\cdot\vec{c}). \quad \ldots (1)$$

$$\text{Now, } \overrightarrow{BD} = \overrightarrow{AD}-\overrightarrow{AB} = \frac{\vec{b}+\vec{c}}{2}-\vec{b} = \frac{\vec{c}-\vec{b}}{2}.$$

Then, this gives, $|\overrightarrow{BD}|^2 = \overrightarrow{BD}\cdot\overrightarrow{BD} = \left(\frac{\vec{c}-\vec{b}}{2}\right)\cdot\left(\frac{\vec{c}-\vec{b}}{2}\right) = \frac{1}{4}(|\vec{b}|^2+|\vec{c}|^2-2\vec{b}\cdot\vec{c})$ ... (2)

Now, adding (1) and (2), we get

Therefore, $|\overrightarrow{AD}|^2+|\overrightarrow{BD}|^2 = \frac{1}{4}(|\vec{b}|^2+|\vec{c}|^2+2\vec{b}\cdot\vec{c})+\frac{1}{4}(|\vec{b}|^2+|\vec{c}|^2-2\vec{b}\cdot c) = \frac{1}{2}(|\vec{b}|^2+|\vec{c}|^2)$

$\Rightarrow$ $|\overrightarrow{AD}|^2+|\overrightarrow{BD}|^2 = \frac{1}{2}(|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2)$.

Hence, $|\overrightarrow{AB}|^2+|\overrightarrow{AC}|^2 = 2(|\overrightarrow{AD}|^2+|\overrightarrow{BD}|^2)$ ■

Example 6.7

Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

Solution

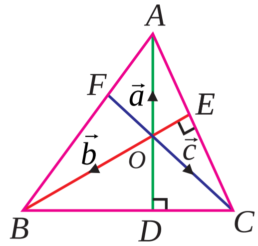


Fig. 6.12

Consider a triangle ABC in which the two altitudes AD and BE intersect at O. Let CO be produced to meet AB at F. We take O as the origin and let $\overrightarrow{OA}=\vec{a}, \overrightarrow{OB}=\vec{b}$ and $\overrightarrow{OC}=\vec{c}$.

Since $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{BC}$, we have $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$, and hence we get $\overrightarrow{OA}\cdot\overrightarrow{BC}=0$. That is, $\vec{a}\cdot(\vec{c}-\vec{b})=0$, which means

$$\vec{a}\cdot\vec{c}-\vec{a}\cdot\vec{b} = 0. \quad \ldots (1)$$

Similarly, since $\overrightarrow{BE}$ is perpendicular to $\overrightarrow{CA}$, we have $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CA}$, and hence we get $\overrightarrow{OB}\cdot\overrightarrow{CA}=0$. That is, $\vec{b}\cdot(\vec{a}-\vec{c})=0$, which means,

$$\vec{a}\cdot\vec{b}-\vec{b}\cdot\vec{c} = 0. \quad \ldots (2)$$

Adding equations (1) and (2), gives $\vec{a}\cdot\vec{c}-\vec{b}\cdot\vec{c}=0$. That is, $\vec{c}\cdot(\vec{a}-\vec{b})=0$.

That is, $\overrightarrow{OC}\cdot\overrightarrow{BA}=0$. Therefore, $\overrightarrow{BA}$ is perpendicular to $\overrightarrow{OC}$ which implies that $\overrightarrow{CF}$ is perpendicular to $\overrightarrow{AB}$. Hence, the perpendicular drawn from C to the side AB passes through O. Thus, the altitudes are concurrent. ■

Example 6.8

In triangle ABC, the points D, E, F are the midpoints of the sides BC, CA, and AB respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}$ (area of ΔABC).

Solution

In triangle ABC, consider A as the origin. Then the position vectors of D, E, F are given by $\dfrac{\overrightarrow{AB}+\overrightarrow{AC}}{2}, \dfrac{\overrightarrow{AC}}{2}, \dfrac{\overrightarrow{AB}}{2}$ respectively. Since $|\overrightarrow{AB}\times\overrightarrow{AC}|$ is the area of the parallelogram formed by the two vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ as adjacent sides, the area of ΔABC is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$. Similarly, considering ΔDEF, we have

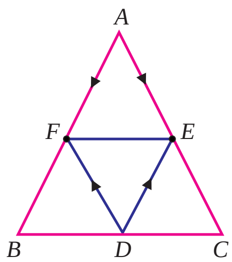


Fig. 6.13

$$\begin{aligned}
\text{the area of } \Delta DEF &= \frac{1}{2}|\overrightarrow{DE}\times\overrightarrow{DF}| \\
&= \frac{1}{2}|(\overrightarrow{AE}-\overrightarrow{AD})\times(\overrightarrow{AF}-\overrightarrow{AD})| \\
&= \frac{1}{2}\left|\frac{\overrightarrow{AB}}{2}\times\frac{\overrightarrow{AC}}{2}\right| \\
&= \frac{1}{4}\left(\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|\right) \\
&= \frac{1}{4}(\text{the area of } \Delta ABC).
\end{aligned}$$

6.3.4 Application of dot and cross product in Physics

Definition 6.2

If $\vec{d}$ is the displacement vector of a particle moved from a point to another point after applying a constant force $\vec{F}$ on the particle, then the **work done by the force** on the particle is $w=\vec{F}\cdot\vec{d}$.

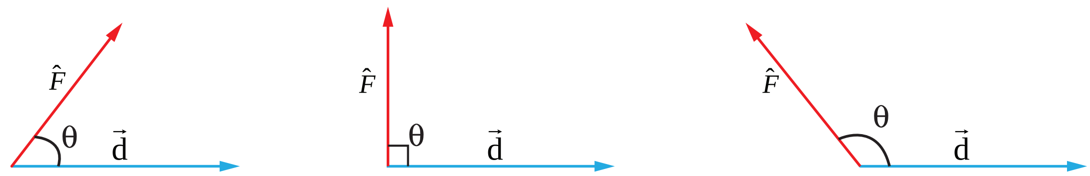


Fig. 6.14

If the force has an acute angle, perpendicular angle, and an obtuse angle, the work done by the force is positive, zero, and negative respectively.

Example 6.9

A particle acted upon by constant forces $2\hat{i}+5\hat{j}+6\hat{k}$ and $-\hat{i}-2\hat{j}-\hat{k}$ is displaced from the point $(4,-3,-2)$ to the point $(6,1,-3)$. Find the total work done by the forces.

Solution

Resultant of the given forces is $\vec{F}=(2\hat{i}+5\hat{j}+6\hat{k})+(-\hat{i}-2\hat{j}-\hat{k})=\hat{i}+3\hat{j}+5\hat{k}$.

Let A and B be the points $(4,-3,-2)$ and $(6,1,-3)$ respectively. Then the displacement vector of the particle is $\vec{d}=\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(6\hat{i}+\hat{j}-3\hat{k})-(4\hat{i}-3\hat{j}-2\hat{k})=2\hat{i}+4\hat{j}-\hat{k}$.

Therefore the work done $w=\vec{F}\cdot\vec{d}=(\hat{i}+3\hat{j}+5\hat{k})\cdot(2\hat{i}+4\hat{j}-\hat{k})=9$ units.

Example 6.10

A particle is acted upon by the forces $3\hat{i}-2\hat{j}+2\hat{k}$ and $2\hat{i}+\hat{j}-\hat{k}$ is displaced from the point $(1,3,-1)$ to the point $(4,-1,\lambda)$. If the work done by the forces is 16 units, find the value of λ.

Solution

Resultant of the given forces is $\vec{F}=(3\hat{i}-2\hat{j}+2\hat{k})+(2\hat{i}+\hat{j}-\hat{k})=5\hat{i}-\hat{j}+\hat{k}$.

The displacement of the particle is given by

$$\vec{d}=(4\hat{i}-\hat{j}+\lambda\hat{k})-(\hat{i}+3\hat{j}-\hat{k})=(3\hat{i}-4\hat{j}+(\lambda+1)\hat{k}).$$

As the work done by the forces is 16 units, we have

$$\vec{F}\cdot\vec{d}=16.$$

That is, $(5\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}-4\hat{j}+(\lambda+1)\hat{k}=16 \Rightarrow \lambda+20=16$.

So, $\lambda=-4$.

Definition 6.3

If a force $\vec{F}$ is applied on a particle at a point with position vector $\vec{r}$, then the **torque** or **moment** on the particle is given by $\vec{t}=\vec{r}\times\vec{F}$. The torque is also called the rotational force.

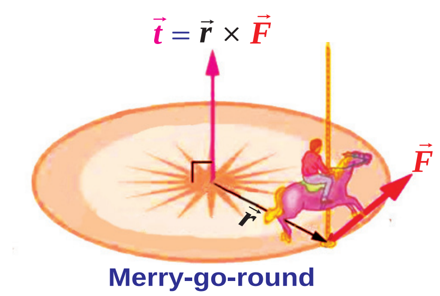


Fig. 6.15

Example 6.11

Find the magnitude and the direction cosines of the torque about the point $(2,0,-1)$ of a force $2\hat{i}+\hat{j}-\hat{k}$, whose line of action passes through the origin.

Solution

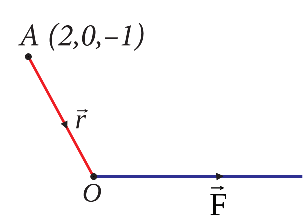


Fig. 6.16

Let A be the point $(2,0,-1)$. Then the position vector of A is $\overrightarrow{OA}=2\hat{i}-\hat{k}$ and therefore $\vec{r}=\overrightarrow{AO}=-2\hat{i}+\hat{k}$.

Then the given force is $\vec{F}=2\hat{i}+\hat{j}-\hat{k}$. So, the torque is

$$\vec{t}=\vec{r}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & 1\\ 2 & 1 & -1\end{vmatrix}=-\hat{i}-2\hat{k}.$$

The magnitude of the torque $=|-\hat{i}-2\hat{k}|=\sqrt{5}$ and the direction cosines of the torque are $-\dfrac{1}{\sqrt{5}},0,-\dfrac{2}{\sqrt{5}}$.

EXERCISE 6.1

1. Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.
2. Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.
3. Prove by vector method that an angle in a semi-circle is a right angle.
4. Prove by vector method that the diagonals of a rhombus bisect each other at right angles.
5. Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.
6. Prove by vector method that the area of the quadrilateral $ABCD$ having diagonals AC and BD is $\frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BD}|$.
7. Prove by vector method that the parallelograms on the same base and between the same parallels are equal in area.
8. If G is the centroid of a ΔABC, prove that (area of ΔGAB) = (area of ΔGBC) = (area of ΔGCA) $=\frac{1}{3}$ (area of ΔABC).
9. Using vector method, prove that $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$.
10. Prove by vector method that $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$,
11. A particle acted on by constant forces $8\hat{i}+2\hat{j}-6\hat{k}$ and $6\hat{i}+2\hat{j}-2\hat{k}$ is displaced from the point $(1,2,3)$ to the point $(5,4,1)$. Find the total work done by the forces.
12. Forces of magnitudes $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $3\hat{i}+4\hat{j}+5\hat{k}$ and $10\hat{i}+6\hat{j}-8\hat{k}$, respectively, act on a particle which is displaced from the point with position vector $4\hat{i}-3\hat{j}-2\hat{k}$ to the point with position vector $6\hat{i}+\hat{j}-3\hat{k}$. Find the work done by the forces.
13. Find the magnitude and direction cosines of the torque of a force represented by $3\hat{i}+4\hat{j}-5\hat{k}$ about the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ acting through a point whose position vector is $4\hat{i}+2\hat{j}-3\hat{k}$.
14. Find the torque of the resultant of the three forces represented by $-3\hat{i}+6\hat{j}-3\hat{k}$, $4\hat{i}-10\hat{j}+12\hat{k}$ and $4\hat{i}+7\hat{j}$ acting at the point with position vector $8\hat{i}-6\hat{j}-4\hat{k}$, about the point with position vector $18\hat{i}+3\hat{j}-9\hat{k}$.

6.4 Scalar triple product

Definition 6.4

For a given set of three vectors $\vec{a},\vec{b}$, and $\vec{c}$, the scalar $(\vec{a}\times\vec{b})\cdot\vec{c}$ is called a **scalar triple product** of $\vec{a},\vec{b},\vec{c}$.

Remark

$\vec{a}\cdot\vec{b}$ is a scalar and so $(\vec{a}\cdot\vec{b})\times\vec{c}$ has no meaning.

Note

Given any three vectors $\vec{a},\vec{b}$ and $\vec{c}$, the following are scalar triple products:

$$(\vec{a}\times\vec{b})\cdot\vec{c},\ (\vec{b}\times\vec{c})\cdot\vec{a},\ (\vec{c}\times\vec{a})\cdot\vec{b},\ \vec{a}\cdot(\vec{b}\times\vec{c}),\ \vec{b}\cdot(\vec{c}\times\vec{a}),\ \vec{c}\cdot(\vec{a}\times\vec{b}),$$

$$(\vec{b}\times\vec{a})\cdot\vec{c},\ (\vec{c}\times\vec{b})\cdot\vec{a},\ (\vec{a}\times\vec{c})\cdot\vec{b},\ \vec{a}\cdot(\vec{c}\times\vec{b}),\ \vec{b}\cdot(\vec{a}\times\vec{c}),\ \vec{c}\cdot(\vec{b}\times\vec{a})$$

Geometrical interpretation of scalar triple product

Geometrically, the absolute value of the scalar triple product $(\vec{a}\times\vec{b})\cdot\vec{c}$ is the volume of the parallelepiped formed by using the three vectors $\vec{a},\vec{b},$ and $\vec{c}$ as co-terminus edges. Indeed, the magnitude of the vector $(\vec{a}\times\vec{b})$ is the area of the parallelogram formed by using $\vec{a}$ and $\vec{b}$; and the direction of the vector $(\vec{a}\times\vec{b})$ is perpendicular to the plane parallel to both $\vec{a}$ and $\vec{b}$.

Therefore, $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ is $|\vec{a}\times\vec{b}||\vec{c}||\cos\theta|$, where θ is the angle between $\vec{a}\times\vec{b}$ and $\vec{c}$ .From Fig. 6.17, we observe that $|\vec{c}||\cos\theta|$ is the height of the parallelepiped formed by using the three vectors as adjacent vectors. Thus, $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ is the volume of the parallelepiped.

The following theorem is useful for computing scalar triple products.

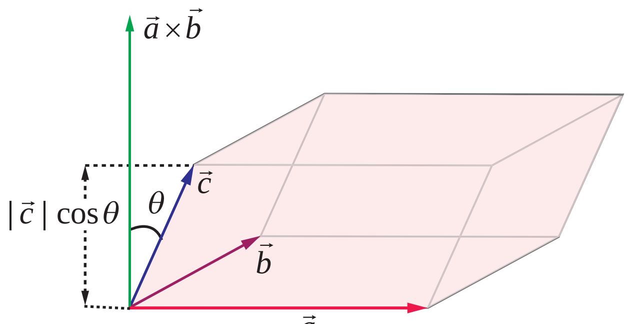


Fig. 6.17

Theorem 6.1

If $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$, then

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Proof

By definition, we have

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\cdot\vec{c}$$

$$= \left[(a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}\right]\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k})$$

$$= (a_2b_3-a_3b_2)c_1+(a_3b_1-a_1b_3)c_2+(a_1b_2-a_2b_1)c_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

which completes the proof of the theorem.

6.4.1 Properties of the scalar triple product

Theorem 6.2

For any three vectors $\vec{a},\vec{b},$ and $\vec{c}$, $(\vec{a}\times\vec{b})\cdot\vec{c}=\vec{a}\cdot(\vec{b}\times\vec{c})$.

Proof

Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$.

$$\text{Then, } \vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{b}\times\vec{c})\cdot\vec{a} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \text{ by } R_1 \leftrightarrow R_3$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \text{ by } R_2 \leftrightarrow R_3$$

$$= (\vec{a}\times\vec{b})\cdot\vec{c}.$$

Hence the theorem is proved. ■

Note

By Theorem 6.2, it follows that, in a scalar triple product, dot and cross can be **interchanged without altering the order of occurrences of the vectors**, by placing the parentheses in such a way that dot lies outside the parentheses, and cross lies between the vectors inside the parentheses. For instance, we have

$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$, since dot and cross can be interchanged.

$= (\vec{b}\times\vec{c})\cdot\vec{a}$, since dot product is commutative.

$= \vec{b}\cdot(\vec{c}\times\vec{a})$, since dot and cross can be interchanged

$= (\vec{c}\times\vec{a})\cdot\vec{b}$, since dot product is commutative

$= \vec{c}\cdot(\vec{a}\times\vec{b})$, since dot and cross can be interchanged

Notation

For any three vectors $\vec{a},\vec{b}$ and $\vec{c}$, the scalar triple product $(\vec{a}\times\vec{b})\cdot\vec{c}$ is denoted by $[\vec{a},\vec{b},\vec{c}]$.

$[\vec{a},\vec{b},\vec{c}]$ is read as **box $\vec{a},\vec{b},\vec{c}$**. For this reason and also because the absolute value of a scalar triple product represents the volume of a box (rectangular parallelepiped),a scalar triple product is also called a **box product**.

Note

(1) $[\vec{a},\vec{b},\vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{b}\times\vec{c})\cdot\vec{a} = \vec{b}\cdot(\vec{c}\times\vec{a}) = [\vec{b},\vec{c},\vec{a}]$

$[\vec{b},\vec{c},\vec{a}] = (\vec{b}\times\vec{c})\cdot\vec{a} = \vec{b}\cdot(\vec{c}\times\vec{a}) = (\vec{c}\times\vec{a})\cdot\vec{b} = \vec{c}\cdot(\vec{a}\times\vec{b}) = [\vec{c},\vec{a},\vec{b}]$.

In other words, $[\vec{a},\vec{b},\vec{c}]=[\vec{b},\vec{c},\vec{a}]=[\vec{c},\vec{a},\vec{b}]$; that is, if the three vectors are permuted in the same cyclic order, the value of the scalar triple product remains the same.

(2) If any two vectors are interchanged in their position in a scalar triple product, then the value of the scalar triple product is (-1) times the original value. More explicitly,

$$[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}]=[\vec{c},\vec{a},\vec{b}]=-[\vec{a},\vec{c},\vec{b}]=-[\vec{c},\vec{b},\vec{a}]=-[\vec{b},\vec{a},\vec{c}].$$

Theorem 6.3

The scalar triple product preserves addition and scalar multiplication. That is,

$$[(\vec{a}+\vec{b}),\vec{c},\vec{d}] = [\vec{a},\vec{c},\vec{d}]+[\vec{b},\vec{c},\vec{d}];$$
$$[\lambda\vec{a},\vec{b},\vec{c}] = \lambda[\vec{a},\vec{b},\vec{c}], \forall\lambda\in\mathbb{R}$$
$$[\vec{a},(\vec{b}+\vec{c}),\vec{d}] = [\vec{a},\vec{b},\vec{d}]+[\vec{a},\vec{c},\vec{d}];$$
$$[\vec{a},\lambda\vec{b},\vec{c}] = \lambda[\vec{a},\vec{b},\vec{c}], \forall\lambda\in\mathbb{R}$$
$$[\vec{a},\vec{b},(\vec{c}+\vec{d})] = [\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{d}];$$
$$[\vec{a},\vec{b},\lambda\vec{c}] = \lambda[\vec{a},\vec{b},\vec{c}], \forall\lambda\in\mathbb{R}.$$

Proof

Using the properties of scalar product and vector product, we get

$$\begin{aligned}[(\vec{a}+\vec{b}),\vec{c},\vec{d}] &= ((\vec{a}+\vec{b})\times\vec{c})\cdot\vec{d}\\ &= (\vec{a}\times\vec{c}+\vec{b}\times\vec{c})\cdot\vec{d}\\ &= (\vec{a}\times\vec{c})\cdot\vec{d}+(\vec{b}\times\vec{c})\cdot\vec{d}\\ &= [\vec{a},\vec{c},\vec{d}]+[\vec{b},\vec{c},\vec{d}]\\ [\lambda\vec{a},\vec{b},\vec{c}] &= ((\lambda\vec{a})\times\vec{b})\cdot\vec{c} = (\lambda(\vec{a}\times\vec{b}))\cdot\vec{c} = \lambda((\vec{a}\times\vec{b})\cdot\vec{c}) = \lambda[\vec{a},\vec{b},\vec{c}].\end{aligned}$$

Using the first statement of this result, we get the following.

$$\begin{aligned}[\vec{a},(\vec{b}+\vec{c}),\vec{d}] &= [(\vec{b}+\vec{c}),\vec{d},\vec{a}] = [\vec{b},\vec{d},\vec{a}]+[\vec{c},\vec{d},\vec{a}]\\ &= [\vec{a},\vec{b},\vec{d}]+[\vec{a},\vec{c},\vec{d}]\\ [\vec{a},\lambda\vec{b},\vec{c}] &= [\lambda\vec{b},\vec{c},\vec{a}] = \lambda[\vec{b},\vec{c},\vec{a}] = \lambda[\vec{a},\vec{b},\vec{c}].\end{aligned}$$

Similarly, the remaining equalities are proved. ■

We have studied about coplanar vectors in XI standard as three nonzero vectors of which, one can be expressed as a linear combination of the other two. Now we use scalar triple product for the characterisation of coplanar vectors.

Theorem 6.4

The scalar triple product of three non-zero vectors is zero if, and only if, the three vectors are coplanar.

Proof

Let $\vec{a},\vec{b},\vec{c}$ be any three non-zero vectors. Then,

$$\begin{aligned}\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=0 &\Leftrightarrow \vec{c} \text{ is perpendicular to } \vec{a}\times\vec{b}\\ &\Leftrightarrow \vec{c} \text{ lies in the plane which is parallel to both } \vec{a} \text{ and } \vec{b}\\ &\Leftrightarrow \vec{a},\vec{b},\vec{c} \text{ are coplanar.}\end{aligned}$$ ■

Theorem 6.5

Three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if, and only if, there exist scalars $r,s,t\in\mathbb{R}$ such that atleast one of them is non-zero and $r\vec{a}+s\vec{b}+t\vec{c}=\vec{0}$.

Proof

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k},\ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then, we have

$\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow \left[\vec{a}, \vec{b}, \vec{c}\right] = 0 \Leftrightarrow \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

$\Leftrightarrow$ there exist scalars $r, s, t \in \mathbb{R}$,

atleast one of them non-zero such that

$a_1 r + a_2 s + a_3 t = 0,\ b_1 r + b_2 s + b_3 t = 0,\ c_1 r + c_2 s + c_3 t = 0$

$\Leftrightarrow$ there exist scalars $r, s, t \in \mathbb{R}$,

atleast one of them non-zero such that $r\vec{a} + s\vec{b} + t\vec{c} = \vec{0}$.

Theorem 6.6

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{p}, \vec{q}, \vec{r}$ are any two systems of three vectors, and if $\vec{p} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $\vec{q} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$, and, $\vec{r} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$, then

$$\left[\vec{p}, \vec{q}, \vec{r}\right] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \left[\vec{a}, \vec{b}, \vec{c}\right].$$

Note

By theorem 6.6, if $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar and

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \neq 0,$$

then the three vectors $\vec{p} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$, $\vec{q} = x_2\vec{a} + y_2\vec{b} + z_2\vec{c}$, and, $\vec{r} = x_3\vec{a} + y_3\vec{b} + z_3\vec{c}$ are also non-coplanar.

Example 6.12

If $\vec{a} = -3\hat{i} - \hat{j} + 5\hat{k},\ \vec{b} = \hat{i} - 2\hat{j} + \hat{k},\ \vec{c} = 4\hat{j} - 5\hat{k}$, find $\vec{a} \cdot (\vec{b} \times \vec{c})$.

Solution: By the defination of scalar triple product of three vectors,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix} = -3.$$

Example 6.13

Find the volume of the parallelepiped whose coterminus edges are given by the vectors $2\hat{i}-3\hat{j}+4\hat{k}, \hat{i}+2\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$.

Solution

We know that the volume of the parallelepiped whose coterminus edges are $\vec{a},\vec{b},\vec{c}$ is given by $|[\vec{a},\vec{b},\vec{c}]|$. Here, $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$, $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$.

Since $[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}=-7$, the volume of the parallelepiped is $|-7|=7$ cubic units.

Example 6.14

Show that the vectors $\hat{i}+2\hat{j}-3\hat{k}$, $2\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{j}-\hat{k}$ are coplanar.

Solution

Here, $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+2\hat{k}$, $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$

We know that $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if $[\vec{a},\vec{b},\vec{c}]=0$. Now, $[\vec{a},\vec{b},\vec{c}]=\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}=0$.

Therefore, the three given vectors are coplanar.

Example 6.15

If $2\hat{i}-\hat{j}+3\hat{k}$, $3\hat{i}+2\hat{j}+\hat{k}$, $\hat{i}+m\hat{j}+4\hat{k}$ are coplanar, find the value of m.

Solution

Since the given three vectors are coplanar, we have $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix}=0 \Rightarrow m=-3$.

Example 6.16

Show that the four points $(6,-7,0)$, $(16,-19,-4)$, $(0,3,-6)$, $(2,-5,10)$ lie on a same plane.

Solution

Let $A=(6,-7,0)$, $B=(16,-19,-4)$, $C=(0,3,-6)$, $D=(2,-5,10)$. To show that the four points A,B,C,D lie on a plane, we have to prove that the three vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are coplanar.

Now, $\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}=(16\hat{i}-19\hat{j}-4\hat{k})-(6\hat{i}-7\hat{j})=10\hat{i}-12\hat{j}-4\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA}=-6\hat{i}+10\hat{j}-6\hat{k}$ and $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=-4\hat{i}+2\hat{j}+10\hat{k}$.

We have $[\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix}=0$.

Therefore, the three vectors $\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}$ are coplanar and hence the four points $A,B,C,$ and D lie on a plane.

Example 6.17

If the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, then prove that the vectors $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are also coplanar.

Solution

Since the vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, we have $[\vec{a},\vec{b},\vec{c}]=0$. Using the properties of the scalar triple product, we get

$$\begin{aligned}[\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}] &= [\vec{a},\vec{b}+\vec{c},\vec{c}+\vec{a}]+[\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}]\\ &= [\vec{a},\vec{b},\vec{c}+\vec{a}]+[\vec{a},\vec{c},\vec{c}+\vec{a}]+[\vec{b},\vec{b},\vec{c}+\vec{a}]+[\vec{b},\vec{c},\vec{c}+\vec{a}]\\ &= [\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{a}]+[\vec{a},\vec{c},\vec{c}]+[\vec{a},\vec{c},\vec{a}]+[\vec{b},\vec{b},\vec{c}]+[\vec{b},\vec{b},\vec{a}]+[\vec{b},\vec{c},\vec{c}]+[\vec{b},\vec{c},\vec{a}]\\ &= [\vec{a},\vec{b},\vec{c}]+[\vec{a},\vec{b},\vec{c}]=2[a,b,c]=0.\end{aligned}$$

Hence the vectors $\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}$ are coplanar. ■

Example 6.18

If $\vec{a},\vec{b},\vec{c}$ are three vectors, prove that $[\vec{a}+\vec{c},\ \vec{a}+\vec{b},\ \vec{a}+\vec{b}+\vec{c}]=[\vec{a},\vec{b},\vec{c}]$.

Solution

Using theorem 6.6, we get

$$\begin{aligned}[\vec{a}+\vec{c},\vec{a}+\vec{b},\ \vec{a}+\vec{b}+\vec{c}] &= \begin{vmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 1 & 1 & 1\end{vmatrix}[\vec{a},\vec{b},\vec{c}]\\ &= [\vec{a},\vec{b},\vec{c}].\end{aligned}$$

■

EXERCISE 6.2

1. If $\vec{a}=\hat{i}-2\hat{j}+3\hat{k},\ \vec{b}=2\hat{i}+\hat{j}-2\hat{k},\ \vec{c}=3\hat{i}+2\hat{j}+\hat{k}$, find $\vec{a}\cdot(\vec{b}\times\vec{c})$.
2. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{i}+14\hat{j}+10\hat{k},\ 14\hat{i}-10\hat{j}-6\hat{k}$ and $2\hat{i}+4\hat{j}-2\hat{k}$.
3. The volume of the parallelepiped whose coterminus edges are $7\hat{i}+\lambda\hat{j}-3\hat{k},\ \hat{i}+2\hat{j}-\hat{k},$ $-3\hat{i}+7\hat{j}+5\hat{k}$ is 90 cubic units. Find the value of λ.
4. If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar vectors represented by concurrent edges of a parallelepiped of volume 4 cubic units, find the value of $(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c})+(\vec{b}+\vec{c})\cdot(\vec{c}\times\vec{a})+(\vec{c}+\vec{a})\cdot(\vec{a}\times\vec{b})$.
5. Find the altitude of a parallelepiped determined by the vectors $\vec{a}=-2\hat{i}+5\hat{j}+3\hat{k},\ \vec{b}=\hat{i}+3\hat{j}-2\hat{k}$ and $\vec{c}=-3\hat{i}+\hat{j}+4\hat{k}$ if the base is taken as the parallelogram determined by $\vec{b}$ and $\vec{c}$.
6. Determine whether the three vectors $2\hat{i}+3\hat{j}+\hat{k},\ \hat{i}-2\hat{j}+2\hat{k}$ and $3\hat{i}+\hat{j}+3\hat{k}$ are coplanar.
7. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=\hat{i}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. If $c_1=1$ and $c_2=2$, find c_3 such that $\vec{a},\vec{b}$ and $\vec{c}$ are coplanar.

8. If $\vec{a}=\hat{i}-\hat{k}, \vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}, \vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$, show that $[\vec{a},\vec{b},\vec{c}]$ depends on neither x nor y.
9. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar, prove that c is the geometric mean of a and b.
10. Let $\vec{a},\vec{b},\vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, show that $[\vec{a},\vec{b},\vec{c}]^2=\frac{1}{4}|\vec{a}|^2\,|\vec{b}|^2$.

6.5 Vector triple product

Definition 6.5

For a given set of three vectors $\vec{a},\vec{b},\vec{c}$, the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is called a **vector triple product**.

Note

Given any three vectors $\vec{a},\vec{b},\vec{c}$ the following are vector triple products :

$(\vec{a}\times\vec{b})\times\vec{c}, (\vec{b}\times\vec{c})\times\vec{a}, (\vec{c}\times\vec{a})\times\vec{b}, \vec{c}\times(\vec{a}\times\vec{b}), \vec{a}\times(\vec{b}\times\vec{c}), \vec{b}\times(\vec{c}\times\vec{a})$

Using the well known properties of the vector product, we get the following theorem.

Theorem 6.7

The vector triple product satisfies the following properties.

(1) $(\vec{a}_1+\vec{a}_2)\times(\vec{b}\times\vec{c}) = \vec{a}_1\times(\vec{b}\times\vec{c})+\vec{a}_2\times(\vec{b}\times\vec{c}), (\lambda\vec{a})\times(\vec{b}\times\vec{c})=\lambda(\vec{a}\times(\vec{b}\times\vec{c})), \lambda\in\mathbb{R}$

(2) $\vec{a}\times((\vec{b}_1+\vec{b}_2)\times\vec{c}) = \vec{a}\times(\vec{b}_1\times\vec{c})+\vec{a}\times(\vec{b}_2\times\vec{c}), \vec{a}\times((\lambda\vec{b})\times\vec{c})=\lambda(\vec{a}\times(\vec{b}\times\vec{c})), \lambda\in\mathbb{R}$

(3) $\vec{a}\times(\vec{b}\times(\vec{c}_1+\vec{c}_2)) = \vec{a}\times(\vec{b}\times\vec{c}_1)+\vec{a}\times(\vec{b}\times\vec{c}_2), \vec{a}\times(\vec{b}\times(\lambda\vec{c}))=\lambda(\vec{a}\times(\vec{b}\times\vec{c})), \lambda\in\mathbb{R}$

Remark

Vector triple product is not associative. This means that $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$, for some vectors $\vec{a},\vec{b},\vec{c}$.

Justification

We take $\vec{a}=\hat{i}, \vec{b}=\hat{i}, \vec{c}=\hat{j}$. Then, $\vec{a}\times(\vec{b}\times\vec{c})=\hat{i}\times(\hat{i}\times\hat{j})=\hat{i}\times\hat{k}=-\hat{j}$ but $(\hat{i}\times\hat{i})\times\hat{j}=\vec{0}\times\hat{j}=\vec{0}$.

Therefore, $\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$.

The following theorem gives a simple formula to evaluate the vector triple product.

Theorem 6.8 (Vector Triple product expansion)

For any three vectors $\vec{a},\vec{b},\vec{c}$ we have $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.

Proof

Let us choose the coordinate axes as follows :

Let x-axis be chosen along the line of action of $\vec{a}$, y-axis be chosen in the plane passing through $\vec{a}$ and parallel to $\vec{b}$, and z-axis be chosen perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Then, we have

$$\vec{a} = a_1\hat{i}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Now, $\vec{a}\times(\vec{b}\times\vec{c}) = a_1\hat{i}\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & 0\\ c_1 & c_2 & c_3\end{vmatrix}$

$$= a_1\hat{i}\times\left(b_2c_3\hat{i} - b_1c_3\hat{j} + (b_1c_2 - b_2c_1)\hat{k}\right)$$
$$= -a_1b_1c_3\hat{k} + a_1(b_2c_1 - b_1c_2)\hat{j} \qquad \text{... (1)}$$
$$(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} = a_1c_1\times\left(b_1\hat{i} + b_2\hat{j}\right) - a_1b_1\left(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right)$$
$$= a_1(b_2c_1 - b_1c_2)\hat{j} - a_1b_1c_3\hat{k} \qquad \text{... (2)}$$

From equations (1) and (2), we get

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (a\cdot\vec{b})\vec{c}$$

■

Note

(1) $\vec{a}\times(\vec{b}\times\vec{c}) = \alpha\vec{b} + \beta\vec{c}$, where $\alpha = \vec{a}\cdot\vec{c}$ and $\beta = -(\vec{a}\cdot\vec{b})$, and so it lies in the plane parallel to $\vec{b}$ and $\vec{c}$.

(2) We also note that

$$(\vec{a}\times\vec{b})\times\vec{c} = -\vec{c}\times(\vec{a}\times\vec{b})$$
$$= -[(\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}]$$
$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

Therefore, $(\vec{a}\times\vec{b})\times\vec{c}$ lies in the plane parallel to $\vec{a}$ and $\vec{b}$.

(3) In $(\vec{a}\times\vec{b})\times\vec{c}$, consider the vectors inside the brackets, call $\vec{b}$ as the middle vector and $\vec{a}$ as the non-middle vector. Similarly, in $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}$ is the middle vector and $\vec{c}$ is the non-middle vector. Then we observe that a vector triple product of these vectors is equal to

$$\lambda \text{ (middle vector) } -\mu \text{ (non-middle vector)}$$

where λ is the dot product of the vectors other than the middle vector and μ is the dot product of the vectors other than the non-middle vector.

6.6 Jacobi's Identity and Lagrange's Identity

Theorem 6.9 (Jacobi's identity)

For any three vectors $\vec{a},\vec{b},\vec{c}$, we have $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = \vec{0}$.

Proof

Using vector triple product expansion, we have

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$
$$\vec{b}\times(\vec{c}\times\vec{a}) = (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a}$$

$$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}.$$

Adding the above equations and using the scalar product of two vectors is commutative, we get $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$.

Theorem 6.10 (Lagrange's identity)

For any four vectors $\vec{a},\vec{b},\vec{c},\vec{d}$, we have $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}$.

Proof

Since dot and cross can be interchanged in a scalar product, we get

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) &= \vec{a}\cdot(\vec{b}\times(\vec{c}\times\vec{d}))\\ &= \vec{a}\cdot((\vec{b}\cdot\vec{d})\vec{c}-(\vec{b}\cdot\vec{c})\vec{d}) \quad \text{(by vector triple product expansion)}\\ &= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})-(\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})\\ &= \begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d}\\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d}\end{vmatrix}\end{aligned}$$

Example 6.19

Prove that $[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$.

Solution

Using the definition of the scalar triple product, we get

$[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}]=(\vec{a}\times\vec{b})\cdot[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$. ... (1)

By treating $(\vec{b}\times\vec{c})$ as the first vector in the vector triple product, we find

$$(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = ((\vec{b}\times\vec{c})\cdot\vec{a})\vec{c}-((\vec{b}\times\vec{c})\cdot\vec{c})\vec{a}=[\vec{a},\vec{b},\vec{c}]\vec{c}.$$

Using this value in (1), we get

$$[\vec{a}\times\vec{b},\vec{b}\times\vec{c},\vec{c}\times\vec{a}] = (\vec{a}\times\vec{b})\cdot([\vec{a},\vec{b},\vec{c}]\vec{c})=[\vec{a},\vec{b},\vec{c}](\vec{a}\times\vec{b})\cdot\vec{c}=[\vec{a},\vec{b},\vec{c}]^2.$$

Example 6.20

Prove that $(\vec{a}\cdot(\vec{b}\times\vec{c}))\vec{a}=(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c})$.

Solution

Treating $(\vec{a}\times\vec{b})$ as the first vector on the right hand side of the given equation and using the vector triple product expansion, we get

$$(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c}) = ((\vec{a}\times\vec{b})\cdot\vec{c})\vec{a}-((\vec{a}\times\vec{b})\cdot\vec{a})\vec{c}=(\vec{a}\cdot(\vec{b}\times\vec{c}))\vec{a}.$$

Example 6.21

For any four vectors $\vec{a},\ \vec{b},\ \vec{c},\ \vec{d}$, we have

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c}-[\vec{a},\vec{b},\vec{c}]\vec{d}=[\vec{a},\vec{c},\vec{d}]\vec{b}-[\vec{b},\vec{c},\vec{d}]\vec{a}.$$

Solution

Taking $\vec{p}=(\vec{a}\times\vec{b})$ as a single vector and using the vector triple product expansion, we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{p}\times(\vec{c}\times\vec{d})$$

$$= (\vec{p}\cdot\vec{d})\vec{c} - (\vec{p}\cdot\vec{c})\vec{d}$$

$$= ((\vec{a}\times\vec{b})\cdot\vec{d})\vec{c} - ((\vec{a}\times\vec{b})\cdot\vec{c})\vec{d} = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$$

Similarly, taking $\vec{q} = \vec{c}\times\vec{d}$, we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{q}$$

$$= (\vec{a}\cdot\vec{q})\vec{b} - (\vec{b}\cdot\vec{q})\vec{a}$$

$$= [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$$ ■

Example 6.22

If $\vec{a} = -2\hat{i}+3\hat{j}-2\hat{k}$, $\vec{b} = 3\hat{i}-\hat{j}+3\hat{k}$, $\vec{c} = 2\hat{i}-5\hat{j}+\hat{k}$, find $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$. State whether they are equal.

Solution

By definition, $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & -2 \\ 3 & -1 & 3 \end{vmatrix} = 7\hat{i} - 7\hat{k}.$$

Then, $$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 0 & -7 \\ 2 & -5 & 1 \end{vmatrix} = -35\hat{i} - 21\hat{j} - 35\hat{k}. \qquad \text{... (1)}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 3 \\ 2 & -5 & 1 \end{vmatrix} = 14\hat{i} + 3\hat{j} - 13\hat{k}.$$

Next, $$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & -2 \\ 14 & 3 & -13 \end{vmatrix} = -33\hat{i} - 54\hat{j} - 48\hat{k}. \qquad \text{... (2)}$$

Therefore, equations (1) and (2) lead to $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$. ■

Example 6.23

If $\vec{a} = \hat{i}-\hat{j}$, $\vec{b} = \hat{i}-\hat{j}-4\hat{k}$, $\vec{c} = 3\hat{j}-\hat{k}$ and $\vec{d} = 2\hat{i}+5\hat{j}+\hat{k}$, verify that

(i) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{b},\vec{d}]\vec{c} - [\vec{a},\vec{b},\vec{c}]\vec{d}$

(ii) $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a},\vec{c},\vec{d}]\vec{b} - [\vec{b},\vec{c},\vec{d}]\vec{a}$

Solution (i)

By definition,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{i} + 4\hat{j},\quad \vec{c}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{i} - 2\hat{j} - 6\hat{k}$$

Then, $$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix} = -24\hat{i}+24\hat{j}-40\hat{k} \quad \text{... (1)}$$

On the other hand, we have

$$[\vec{a},\vec{b},\vec{d}]\vec{c}-[\vec{a},\vec{b},\vec{c}]\vec{d} = 28(3\hat{j}-\hat{k})-12(2\hat{i}+5\hat{j}+\hat{k}) = -24\hat{i}+24\hat{j}-40\hat{k} \quad \text{... (2)}$$

Therefore, from equations (1) and (2), identity (i) is verified.
The verification of identity (ii) is left as an exercise to the reader.

EXERCISE 6.3

1. If $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$, $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$, find (i) $(\vec{a}\times\vec{b})\times\vec{c}$ (ii) $\vec{a}\times(\vec{b}\times\vec{c})$.
2. For any vector $\vec{a}$, prove that $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$.
3. Prove that $[\vec{a}-\vec{b},\ \vec{b}-\vec{c},\ \vec{c}-\vec{a}]=0$.
4. If $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=3\hat{i}+5\hat{j}+2\hat{k}$, $\vec{c}=-\hat{i}-2\hat{j}+3\hat{k}$, verify that

   (i) $(\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}$ (ii) $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$
5. $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}-4\hat{k}$, $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ then find the value of $(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{c})$.
6. If $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar vectors, show that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=\vec{0}$.
7. If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}+\hat{k}$, $\vec{c}=3\hat{i}+2\hat{j}+\hat{k}$ and $\vec{a}\times(\vec{b}\times\vec{c})=l\vec{a}+m\vec{b}+n\vec{c}$, find the values of l,m,n.
8. If $\hat{a},\hat{b},\hat{c}$ are three unit vectors such that $\hat{b}$ and $\hat{c}$ are non-parallel and $\hat{a}\times(\hat{b}\times\hat{c})=\frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$.

6.7 Application of Vectors to 3-Dimensional Geometry

Vectors provide an elegant approach to study straight lines and planes in three dimension. All straight lines and planes are subsets of $\mathbb{R}^3$. For brevity, we shall call a straight line simply as line. A plane is a surface which is understood as a set P of points in $\mathbb{R}^3$ such that , if $A, B,$ and C are any three non-collinear points of P, then the line passing through any two of them is a subset of P. Two planes are said to be intersecting if they have at least one point in common and at least one point which lies on one plane but not on the other. Two planes are said to be coincident if they have exactly the same points. Two planes are said to be parallel but not coincident if they have no point in common. Similarly, a straight line can be understood as the set of points common to two intersecting planes. In this section, we obtain vector and Cartesian equations of straight line and plane by applying vector methods. By a vector form of equation of a geometrical object, we mean an equation which is satisfied by the position vector of every point of the object. The equation may be a vector equation or a scalar equation.

6.7.1 Different forms of equation of a straight line

A straight line can be uniquely fixed if

- a point on the straight line and the direction of the straight line are given
- two points on the straight line are given

We find equations of a straight line in vector and Cartesian form. To find the equation of a straight line in vector form, an arbitrary point P with position vector $\vec{r}$ on the straight line is taken and a relation satisfied by $\vec{r}$ is obtained by using the given conditions. This relation is called the vector equation of the straight line. A vector equation of a straight line may or may not involve parameters. If a vector equation involves parameters, then it is called a **vector equation in parametric form**. If no parameter is involved, then the equation is called a **vector equation in non – parametric form**.

6.7.2 A point on the straight line and the direction of the straight line are given

(a) Parametric form of vector equation

Theorem 6.11

The vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.

Proof

If $\vec{a}$ is the position vector of a given point A and $\vec{r}$ is the position vector of an arbitrary point P on the straight line, then $\overrightarrow{AP} = \vec{r} - \vec{a}$.

Since $\overrightarrow{AP}$ is parallel to $\vec{b}$, we have

$$\vec{r} - \vec{a} = t\vec{b},\ t \in \mathbb{R} \qquad \dots (1)$$

$$\text{or} \quad \vec{r} = \vec{a} + t\vec{b},\ t \in \mathbb{R} \qquad \dots (2)$$

This is the vector equation of the straight line in parametric form.

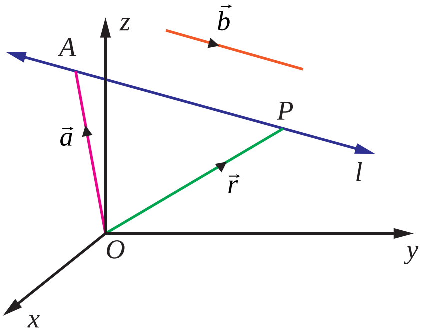


Fig. 6.18

Remark

The position vector of any point on the line is taken as $\vec{a} + t\vec{b}$.

(b) Non-parametric form of vector equation

Since $\overrightarrow{AP}$ is parallel to $\vec{b}$, we have $\overrightarrow{AP} \times \vec{b} = \vec{0}$

That is, $(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$.

This is known as the **vector equation** of the straight line in **non-parametric form**.

(c) Cartesian equation

Suppose P is (x, y, z), A is (x_1, y_1, z_1) and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then, substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ in (1) and comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get

$$x - x_1 = tb_1,\ y - y_1 = tb_2,\ z - z_1 = tb_3 \qquad \dots (4)$$

Conventionally (4) can be written as

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3} \qquad \dots (5)$$

which are called the **Cartesian equations or symmetric equations** of a straight line passing through the point (x_1, y_1, z_1) and parallel to a vector with direction ratios b_1, b_2, b_3.

Remark

(i) Every point on the line (5) is of the form $(x_1 + tb_1, y_1 + tb_2, z_1 + tb_3)$, where $t \in \mathbb{R}$.

(ii) Since the direction cosines of a line are proportional to direction ratios of the line, if l, m, n are the direction cosines of the line, then the Cartesian equations of the line are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

(iii) In (5), if any one or two of b_1, b_2, b_3 are zero, it does not mean that we are dividing by zero. But it means that the corresponding numerator is zero. For instance, If $b_1 \neq 0$, $b_2 \neq 0$ and $b_3 = 0$, then

$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{0}$ should be written as $\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2}, z - z_1 = 0$.

(iv) We know that the direction cosines of x - axis are $1, 0, 0$. Therefore, the equations of x-axis are

$$\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0} \text{ or } x = t, y = 0, z = 0, \text{ where } t \in \mathbb{R}.$$

Similarly the equations of y-axis and z-axis are given by $\frac{x - 0}{0} = \frac{y - 0}{1} = \frac{z - 0}{0}$ and $\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}$ respectively.

6.7.3 Straight Line passing through two given points

(a) Parametric form of vector equation

Theorem 6.12

The parametric form of vector equation of a line passing through two given points whose position vectors are $\vec{a}$ and $\vec{b}$ respectively is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a}), t \in \mathbb{R}$.

(b) Non-parametric form of vector equation

The above equation can be written equivalently in non-parametric form of vector equation as

$$(\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = \vec{0}$$

(c) Cartesian form of equation

Suppose P is (x, y, z), A is (x_1, y_1, z_1) and B is (x_2, y_2, z_2). Then substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ in theorem 6.12 and comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$, we get $x - x_1 = t(x_2 - x_1), y - y_1 = t(y_2 - y_1), z - z_1 = t(z_2 - z_1)$ and so the Cartesian equations of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

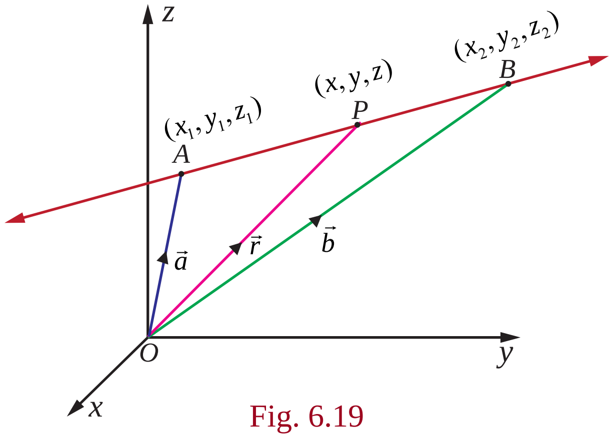


Fig. 6.19

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}.$$

From the above equation, we observe that the direction ratios of a line passing through two given points (x_1,y_1,z_1) and (x_2,y_2,z_2) are given by $x_2-x_1, y_2-y_1, z_2-z_1$, which are also given by any three numbers proportional to them and in particular $x_1-x_2, y_1-y_2, z_1-z_2$.

Example 6.24

A straight line passes through the point $(1,2,-3)$ and parallel to $4\hat{i}+5\hat{j}-7\hat{k}$. Find (i) vector equation in parametric form (ii) vector equation in non-parametric form (iii) Cartesian equations of the straight line.

Solution

The required line passes through $(1,2,-3)$. So, the position vector of the point is $\hat{i}+2\hat{j}-3\hat{k}$.

Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=4\hat{i}+5\hat{j}-7\hat{k}$. Then, we have

(i) vector equation of the required straight line in parametric form is $\vec{r}=\vec{a}+t\vec{b},\ t\in\mathbb{R}$.

Therefore, $\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+t(4\hat{i}+5\hat{j}-7\hat{k}),\ t\in\mathbb{R}$.

(ii) vector equation of the required straight line in non-parametric form is $(\vec{r}-\vec{a})\times\vec{b}=\vec{0}$.

Therefore, $(\vec{r}-(\hat{i}+2\hat{j}-3\hat{k}))\times(4\hat{i}+5\hat{j}-7\hat{k})=\vec{0}$.

(iii) Cartesian equations of the required line are $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$.

Here, $(x_1,y_1,z_1)=(1,2,-3)$ and direction ratios of the required line are proportional to $4,5,-7$. Therefore, Cartesian equations of the straight line are $\dfrac{x-1}{4}=\dfrac{y-2}{5}=\dfrac{z+3}{-7}$. ■

Example 6.25

The vector equation in parametric form of a line is $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+t(2\hat{i}-\hat{j}+3\hat{k})$. Find (i) the direction cosines of the straight line (ii) vector equation in non-parametric form of the line (iii) Cartesian equations of the line.

Solution

Comparing the given equation with equation of a straight line $\vec{r}=\vec{a}+t\vec{b}$, we have $\vec{a}=3\hat{i}-2\hat{j}+6\hat{k}$ and $\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$. Therefore,

(i) If $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then direction ratios of the straight line are b_1,b_2,b_3. Therefore, direction ratios of the given straight line are proportional to $2,-1,3$, and hence the direction cosines of the given straight line are $\dfrac{2}{\sqrt{14}},\dfrac{-1}{\sqrt{14}},\dfrac{3}{\sqrt{14}}$.

(ii) vector equation of the straight line in non-parametric form is given by $(\vec{r}-\vec{a})\times\vec{b}=\vec{0}$.

Therefore, $(\vec{r}-(3\hat{i}-2\hat{j}+6\hat{k}))\times(2\hat{i}-\hat{j}+3\hat{k})=\vec{0}$.

(iii) Here $(x_1,y_1,z_1)=(3,-2,6)$ and the direction ratios are proportional to $2,-1,3$.

Therefore, Cartesian equations of the straight line are $\dfrac{x-3}{2}=\dfrac{y+2}{-1}=\dfrac{z-6}{3}$. ■

Example 6.26

Find the vector equation in parametric form and Cartesian equations of the line passing through $(-4,2,-3)$ and is parallel to the line $\dfrac{-x-2}{4}=\dfrac{y+3}{-2}=\dfrac{2z-6}{3}$.

Solution

Rewriting the given equations as $\dfrac{x+2}{-4}=\dfrac{y+3}{-2}=\dfrac{z-3}{3/2}$ and comparing with $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$,

we have $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=-4\hat{i}-2\hat{j}+\dfrac{3}{2}\hat{k}=-\dfrac{1}{2}(8\hat{i}+4\hat{j}-3\hat{k})$. Clearly, $\vec{b}$ is parallel to the vector $8\hat{i}+4\hat{j}-3\hat{k}$. Therefore, a vector equation of the required straight line passing through the given point $(-4,2,-3)$ and parallel to the vector $8\hat{i}+4\hat{j}-3\hat{k}$ in parametric form is

$$\vec{r} = (-4\hat{i}+2\hat{j}-3\hat{k})+t(8\hat{i}+4\hat{j}-3\hat{k}),\ t\in\mathbb{R}.$$

Therefore, Cartesian equations of the required straight line are given by

$$\frac{x+4}{8}=\frac{y-2}{4}=\frac{z+3}{-3}.$$

Example 6.27

Find the vector equation in parametric form and Cartesian equations of a straight passing through the points $(-5,7,-4)$ and $(13,-5,2)$. Find the point where the straight line crosses the xy-plane.

Solution

The straight line passes through the points $(-5,7,-4)$ and $(13,-5,2)$, and therefore, direction ratios of the straight line joining these two points are $18,-12,6$. That is $3,-2,1$.

So, the straight line is parallel to $3\hat{i}-2\hat{j}+\hat{k}$. Therefore,

- Required vector equation of the straight line in parametric form is $\vec{r}=(-5\hat{i}+7\hat{j}-4\hat{k})+t(3\hat{i}-2\hat{j}+\hat{k})$ or $\vec{r}=(13\hat{i}-5\hat{j}+2\hat{k})+s(3\hat{i}-2\hat{j}+\hat{k})$ where $s,t\in\mathbb{R}$.
- Required cartesian equations of the straight line are $\dfrac{x+5}{3}=\dfrac{y-7}{-2}=\dfrac{z+4}{1}$ or $\dfrac{x-13}{3}=\dfrac{y+5}{-2}=\dfrac{z-2}{1}$.

An arbitrary point on the straight line is of the form

$$(3t-5,-2t+7,\ t-4)\ \text{or}\ (3s+13,-2s-5,\ s+2)$$

Since the straight line crosses the xy-plane, the z-coordinate of the point of intersection is zero. Therefore, we have $t-4=0$, that is, $t=4$, and hence the straight line crosses the xy-plane at $(7,-1,0)$.

Example 6.28

Find the angle made by the straight line $\dfrac{x+3}{2}=\dfrac{y-1}{2}=-z$ with coordinate axes.

Solution

If $\hat{b}$ is a unit vector parallel to the given line, then $\hat{b}=\dfrac{2\hat{i}+2\hat{j}-\hat{k}}{|2\hat{i}+2\hat{j}-\hat{k}|}=\dfrac{1}{3}(2\hat{i}+2\hat{j}-\hat{k})$. Therefore, from the definition of direction cosines of $\hat{b}$, we have

$$\cos\alpha = \frac{2}{3},\ \cos\beta = \frac{2}{3},\ \cos\gamma = -\frac{1}{3},$$

where α, β, γ are the angles made by $\hat{b}$ with the positive x-axis, positive y-axis, and positive z-axis, respectively. As the angle between the given straight line with the coordinate axes are same as the angles made by $\hat{b}$ with the coordinate axes, we have $\alpha = \cos^{-1}\left(\frac{2}{3}\right)$, $\beta = \cos^{-1}\left(\frac{2}{3}\right)$, $\gamma = \cos^{-1}\left(\frac{-1}{3}\right)$, respectively.

6.7.4 Angle between two straight lines

(a) Vector form

The acute angle between two given straight lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is same as that of the angle between $\vec{b}$ and $\vec{d}$. So, $\cos\theta = \frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}$ or $\theta = \cos^{-1}\left(\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right)$.

Remark

(i) The two given lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ are parallel

$\Leftrightarrow \theta = 0 \Leftrightarrow \cos\theta = 1 \Leftrightarrow |\vec{b}\cdot\vec{d}| = |\vec{b}|\,|\vec{d}|$.

(ii) The two given lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ are parallel if, and only if $\vec{b} = \lambda\vec{d}$, for some scalar λ.

(iii) The two given lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ are perpendicular if, and only if $\vec{b}\cdot\vec{d} = 0$.

(b) Cartesian form

If two lines are given in Cartesian form as $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$, then the acute angle θ between the two given lines is given by

$$\theta = \cos^{-1}\left(\frac{|b_1d_1 + b_2d_2 + b_3d_3|}{\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{d_1^2 + d_2^2 + d_3^2}}\right)$$

Remark

(i) The two given lines with direction ratios b_1, b_2, b_3 and d_1, d_2, d_3 are parallel if, and only if $\frac{b_1}{d_1} = \frac{b_2}{d_2} = \frac{b_3}{d_3}$.

(ii) The two given lines with direction ratios b_1, b_2, b_3 and d_1, d_2, d_3 are perpendicular if and only if $b_1d_1 + b_2d_2 + b_3d_3 = 0$.

(iii) If the direction cosines of two given straight lines are l_1, m_1, n_1 and l_2, m_2, n_2, then the angle between the two given straight lines is $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$.

Example 6.29

Find the acute angle between the lines $\vec{r} = (\hat{i} + 2\hat{j} + 4\hat{k}) + t(2\hat{i} + 2\hat{j} + \hat{k})$ and the straight line passing through the points $(5,1,4)$ and $(9,2,12)$.

Solution

We know that the line $\vec{r} = (\hat{i} + 2\hat{j} + 4\hat{k}) + t(2\hat{i} + 2\hat{j} + \hat{k})$ is parallel to the vector $2\hat{i} + 2\hat{j} + \hat{k}$.

Direction ratios of the straight line joining the two given points $(5,1,4)$ and $(9,2,12)$ are $4,1,8$ and hence this line is parallel to the vector $4\hat{i} + \hat{j} + 8\hat{k}$.

Therefore, the acute angle between the given two straight lines is

$$\theta = \cos^{-1}\left(\frac{|\vec{b}\cdot\vec{d}\,|}{|\vec{b}\,||\vec{d}\,|}\right), \text{ where } \vec{b} = 2\hat{i} + 2\hat{j} + \hat{k} \text{ and } \vec{d} = 4\hat{i} + \hat{j} + 8\hat{k}.$$

$$\text{Therefore, } \theta = \cos^{-1}\left(\frac{|(2\hat{i} + 2\hat{j} + \hat{k})\cdot(4\hat{i} + \hat{j} + 8\hat{k})|}{|2\hat{i} + 2\hat{j} + \hat{k}\,||\,4\hat{i} + \hat{j} + 8\hat{k}\,|}\right) = \cos^{-1}\left(\frac{2}{3}\right).$$

Example 6.30

Find the acute angle between the straight lines $\frac{x-4}{2} = \frac{y}{1} = \frac{z+1}{-2}$ and $\frac{x-1}{4} = \frac{y+1}{-4} = \frac{z-2}{2}$ and state whether they are parallel or perpendicular.

Solution

Comparing the given lines with the general Cartesian equations of straight lines,

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3} \text{ and } \frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$$

we find $(b_1, b_2, b_3) = (2,1,-2)$ and $(d_1, d_2, d_3) = (4,-4,2)$. Therefore, the acute angle between the two straight lines is

$$\theta = \cos^{-1}\left(\frac{|(2)(4) + (1)(-4) + (-2)(2)|}{\sqrt{2^2 + 1^2 + (-2)^2}\sqrt{4^2 + (-4)^2 + 2^2}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

Thus the two straight lines are perpendicular.

Example 6.31

Show that the straight line passing through the points $A(6,7,5)$ and $B(8,10,6)$ is perpendicular to the straight line passing through the points $C(10,2,-5)$ and $D(8,3,-4)$.

Solution

The straight line passing through the points $A(6,7,5)$ and $B(8,10,6)$ is parallel to the vector $\vec{b} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\hat{i} + 3\hat{j} + \hat{k}$ and the straight line passing through the points $C(10,2,-5)$ and $D(8,3,-4)$ is parallel to the vector $\vec{d} = \overrightarrow{CD} = -2\hat{i} + \hat{j} + \hat{k}$. Therefore, the angle between the two straight lines is the angle between the two vectors $\vec{b}$ and $\vec{d}$. Since

$$\vec{b}\cdot\vec{d} = (2\hat{i} + 3\hat{j} + \hat{k})\cdot(-2\hat{i} + \hat{j} + \hat{k}) = 0.$$

the two vectors are perpendicular, and hence the two straight lines are perpendicular.

Aliter

We find that direction ratios of the straight line joining the points $A(6,7,5)$ and $B(8,10,6)$ are $(b_1, b_2, b_3) = (2,3,1)$ and direction ratios of the line joining the points $C(10,2,-5)$ and $D(8,3,-4)$ are $(d_1, d_2, d_3) = (-2,1,1)$. Since $b_1d_1 + b_2d_2 + b_3d_3 = (2)(-2)+(3)(1)+(1)(1) = 0$, the two straight lines are perpendicular. ■

Example 6.32

Show that the lines $\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ and $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$ are parallel.

Solution

We observe that the straight line $\frac{x-1}{4} = \frac{2-y}{6} = \frac{z-4}{12}$ is parallel to the vector $4\hat{i} - 6\hat{j} + 12\hat{k}$ and the straight line $\frac{x-3}{-2} = \frac{y-3}{3} = \frac{5-z}{6}$ is parallel to the vector $-2\hat{i} + 3\hat{j} - 6\hat{k}$.

Since $4\hat{i} - 6\hat{j} + 12\hat{k} = -2(-2\hat{i} + 3\hat{j} - 6\hat{k})$, the two vectors are parallel, and hence the two straight lines are parallel. ■

EXERCISE 6.4

1. Find the non-parametric form of vector equation and Cartesian equations of the straight line passing through the point with position vector $4\hat{i} + 3\hat{j} - 7\hat{k}$ and parallel to the vector $2\hat{i} - 6\hat{j} + 7\hat{k}$.
2. Find the parametric form of vector equation and Cartesian equations of the straight line passing through the point $(-2,3,4)$ and parallel to the straight line $\frac{x-1}{-4} = \frac{y+3}{5} = \frac{8-z}{6}$.
3. Find the points where the straight line passes through $(6,7,4)$ and $(8,4,9)$ cuts the xz and yz planes.
4. Find the direction cosines of the straight line passing through the points $(5,6,7)$ and $(7,9,13)$. Also, find the parametric form of vector equation and Cartesian equations of the straight line passing through two given points.
5. Find the acute angle between the following lines.
   (i) $\vec{r} = (4\hat{i} - \hat{j}) + t(\hat{i} + 2\hat{j} - 2\hat{k}),\ \vec{r} = (\hat{i} - 2\hat{j} + 4\hat{k}) + s(-\hat{i} - 2\hat{j} + 2\hat{k})$
   (ii) $\frac{x+4}{3} = \frac{y-7}{4} = \frac{z+5}{5},\ \vec{r} = 4\hat{k} + t(2\hat{i} + \hat{j} + \hat{k})$.
   (iii) $2x = 3y = -z$ and $6x = -y = -4z$.
6. The vertices of ΔABC are $A(7,2,1), B(6,0,3)$, and $C(4,2,4)$. Find $\angle ABC$.
7. If the straight line joining the points $(2,1,4)$ and $(a-1,4,-1)$ is parallel to the line joining the points $(0,2,b-1)$ and $(5,3,-2)$, find the values of a and b.
8. If the straight lines $\frac{x-5}{5m+2} = \frac{2-y}{5} = \frac{1-z}{-1}$ and $x = \frac{2y+1}{4m} = \frac{1-z}{-3}$ are perpendicular to each other, find the value of m.
9. Show that the points $(2,3,4), (-1,4,5)$ and $(8,1,2)$ are collinear.

6.7.5 Point of intersection of two straight lines

If $\frac{x-x_1}{a_1}=\frac{y-y_1}{a_2}=\frac{z-z_1}{a_3}$ and $\frac{x-x_2}{b_1}=\frac{y-y_2}{b_2}=\frac{z-z_2}{b_3}$ are two lines, then every point on the line is of the form $(x_1+sa_1, y_1+sa_2, z_1+sa_3)$ and $(x_2+tb_1, y_2+tb_2, z_2+tb_3)$ respectively. If the lines are intersecting, then there must be a common point. So, at the point of intersection, for some values of s and t, we have

$$(x_1+sa_1, y_1+sa_2, z_1+sa_3) = (x_2+tb_1, y_2+tb_2, z_2+tb_3)$$

$$\text{Therefore,}\quad x_1+sa_1 = x_2+tb_1, y_1+sa_2=y_2+tb_2, z_1+sa_3=z_2+tb_3$$

By solving any two of the above three equations, we obtain the values of s and t. If s and t satisfy the remaining equation, the lines are intersecting lines. Otherwise the lines are non-intersecting . Substituting the value of s, (or by substituting the value of t), we get the point of intersection of two lines.

If the equations of straight lines are given in vector form, write them in cartesian form and proceed as above to find the point of intersection.

Example 6.33

Find the point of intersection of the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-4}{5}=\frac{y-1}{2}=z$.

Solution

Every point on the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=s$ (say) is of the form $(2s+1, 3s+2, 4s+3)$ and every point on the line $\frac{x-4}{5}=\frac{y-1}{2}=z=t$ (say) is of the form $(5t+4, 2t+1, t)$. So, at the point of intersection, for some values of s and t, we have

$$(2s+1, 3s+2, 4s+3) = (5t+4, 2t+1, t)$$

Therefore, $2s-5t=3, 3s-2t=-1$ and $4s-t=-3$. Solving the first two equations we get $t=-1, s=-1$. These values of s and t satisfy the third equation. Therefore, the given lines intersect. Substituting, these values of t or s in the respective points, the point of intersection is $(-1,-1,-1)$.

6.7.6 Shortest distance between two straight lines

We have just explained how the point of intersection of two lines are found and we have also studied how to determine whether the given two lines are parallel or not.

Definition 6.6

Two lines are said to be **coplanar** if they lie in the same plane.

Note

If two lines are either parallel or intersecting, then they are coplanar.

Definition 6.7

Two lines in space are called **skew lines** if they are not parallel and do not intersect

Note

If two lines are skew lines, then they are non coplanar.

If the lines are not parallel and intersect, the distance between them is zero. If they are parallel and non-intersecting, the distance is determined by the length of the line segment perpendicular to both the parallel lines. In the same way, the shortest distance between two skew lines is defined as the length of the line segment perpendicular to both the skew lines. Two lines will either be parallel or skew.

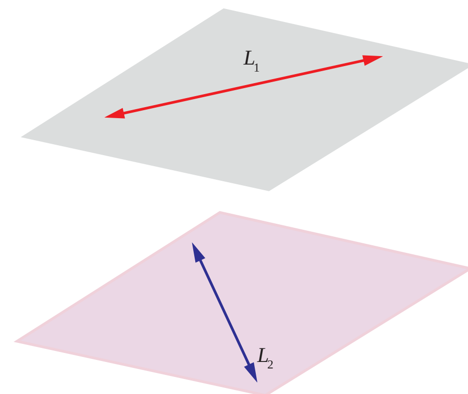


Fig. 6.20

Theorem 6.13

The shortest distance between the two parallel lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{b}$ is given by $d = \dfrac{|(\vec{c} - \vec{a}) \times \vec{b}|}{|\vec{b}|}$, where $|\vec{b}| \neq 0$.

Proof

The given two parallel lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{b}$ are denoted by L_1 and L_2 respectively. Let A and B be the points on L_1 and L_2 whose position vectors are $\vec{a}$ and $\vec{c}$ respectively. The two given lines are parallel to $\vec{b}$.

Let AD be a perpendicular to the two given lines. If θ is the acute angle between $\overrightarrow{AB}$ and $\vec{b}$, then

$$\sin\theta = \frac{|\overrightarrow{AB} \times \vec{b}|}{|\overrightarrow{AB}|\,|\vec{b}|} = \frac{|(\vec{c} - \vec{a}) \times \vec{b}|}{|\vec{c} - \vec{a}|\,|\vec{b}|} \qquad \text{... (1)}$$

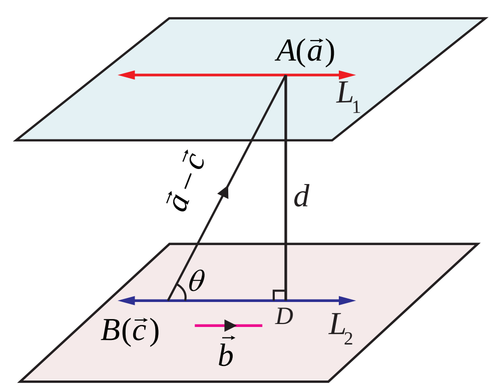


Fig. 6.21

But, from the right angle triangle ABD,

$$\sin\theta = \frac{d}{AB} = \frac{d}{|\overrightarrow{AB}|} = \frac{d}{|\vec{c} - \vec{a}|} \qquad \text{... (2)}$$

From (1) and (2), we have $d = \dfrac{|(\vec{c} - \vec{a}) \times \vec{b}|}{|\vec{b}|}$, where $|\vec{b}| \neq 0$.

Theorem 6.14

The shortest distance between the two skew lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is given by

$$\delta = \frac{|(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|}, \text{ where } |\vec{b} \times \vec{d}| \neq 0$$

Proof

The two skew lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ are denoted by L_1 and L_2 respectively.

Let A and C be the points on L_1 and L_2 with position vectors $\vec{a}$ and $\vec{c}$ respectively.

From the given equations of skew lines, we observe that L_1 is parallel to the vector $\vec{b}$ and L_2 is parallel to the vector $\vec{d}$. So, $\vec{b}\times\vec{d}$ is perpendicular to the lines L_1 and L_2.

Let SD be the line segment perpendicular to both the lines L_1 and L_2. Then the vector $\overrightarrow{SD}$ is perpendicular to the vectors $\vec{b}$ and $\vec{d}$ and therefore it is parallel to the vector $\vec{b}\times\vec{d}$.

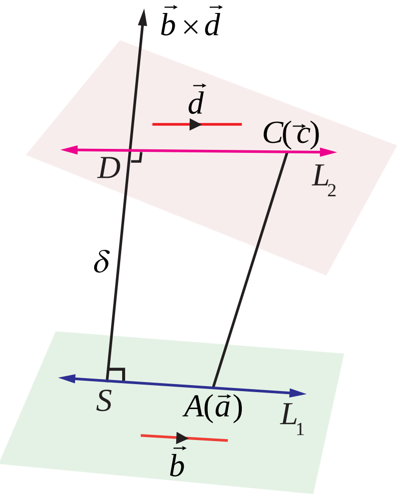


Fig. 6.22

So, $\dfrac{\vec{b}\times\vec{d}}{|\vec{b}\times\vec{d}|}$ is a unit vector in the direction of $\overrightarrow{SD}$. Then, the shortest distance $|\overrightarrow{SD}|$ is the absolute value of the projection of $\overrightarrow{AC}$ on $\overrightarrow{SD}$. That is,

$$\delta = |\overrightarrow{SD}| = |\overrightarrow{AC}\,.\,(\text{Unit vector in the direction of } \overrightarrow{SD})| = \left|(\vec{c}-\vec{a})\cdot\frac{\vec{b}\times\vec{d}}{|\vec{b}\times\vec{d}|}\right|$$

$$\delta = \frac{|(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|}, \text{ where } |\vec{b}\times\vec{d}| \neq 0.$$

Remark

(i) It follows from theorem (6.14) that two straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ intersect each other (that is, coplanar) if $(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=0$.

(2) If two lines $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$ and $\dfrac{x-x_2}{d_1}=\dfrac{y-y_2}{d_2}=\dfrac{z-z_2}{d_3}$ intersect each other (that is, coplanar), then we have

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Example 6.34

Find the parametric form of vector equation of a straight line passing through the point of intersection of the straight lines $\vec{r}=(\hat{i}+3\hat{j}-\hat{k})+t(2\hat{i}+3\hat{j}+2\hat{k})$ and $\dfrac{x-2}{1}=\dfrac{y-4}{2}=\dfrac{z+3}{4}$, and perpendicular to both straight lines.

Solution

The Cartesian equations of the straight line $\vec{r}=(\hat{i}+3\hat{j}-\hat{k})+t(2\hat{i}+3\hat{j}+2\hat{k})$ is

$$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}=s \ \text{(say)}$$

Then any point on this line is of the form $(2s+1,\ 3s+3,\ 2s-1)$ ... (1)

The Cartesian equation of the second line is $\dfrac{x-2}{1}=\dfrac{y-4}{2}=\dfrac{z+3}{4}=t$ (say)

Then any point on this line is of the form $(t+2,\ 2t+4,\ 4t-3)$ ... (2)

If the given lines intersect, then there must be a common point. Therefore, for some $s, t \in \mathbb{R}$, we have $(2s+1,\ 3s+3,\ 2s-1) = (t+2,\ 2t+4, 4t-3)$.

Equating the coordinates of x, y and z we get

$$2s - t = 1,\ 3s - 2t = 1 \text{ and } s - 2t = -1.$$

Solving the first two of the above three equations, we get $s = 1$ and $t = 1$. These values of s and t satisfy the third equation. So, the lines are intersecting.

Now, using the value of s in (1) or the value of t in (2), the point of intersection $(3,6,1)$ of these two straight lines is obtained.

If we take $\vec{b} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{d} = \hat{i} + 2\hat{j} + 4\hat{k}$, then $\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 8\hat{i} - 6\hat{j} + \hat{k}$ is a vector perpendicular to both the given straight lines. Therefore, the required straight line passing through $(3,6,1)$ and perpendicular to both the given straight lines is the same as the straight line passing through $(3,6,1)$ and parallel to $8\hat{i} - 6\hat{j} + \hat{k}$. Thus, the equation of the required straight line is

$$\vec{r} = (3\hat{i} + 6\hat{j} + \hat{k}) + m(8\hat{i} - 6\hat{j} + \hat{k}),\ m \in \mathbb{R}.$$

Example 6.35

Determine whether the pair of straight lines $\vec{r} = (2\hat{i} + 6\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 4\hat{k})$, $\vec{r} = (2\hat{j} - 3\hat{k}) + s(\hat{i} + 2\hat{j} + 3\hat{k})$ are parallel. Find the shortest distance between them.

Solution

Comparing the given two equations with

$$\vec{r} = \vec{a} + s\vec{b} \text{ and } \vec{r} = \vec{c} + s\vec{d},$$

we have $\vec{a} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{c} = 2\hat{j} - 3\hat{k}$, $\vec{d} = \hat{i} + 2\hat{j} + 3\hat{k}$

Clearly, $\vec{b}$ is not a scalar multiple of $\vec{d}$. So, the two vectors are not parallel and hence the two lines are not parallel.

The shortest distance between the two straight lines is given by

$$\delta = \frac{|(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|}$$

$$\text{Now, } \vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$$

$$\text{So, } (\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (-2\hat{i} - 4\hat{j} - 6\hat{k}) \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0.$$

Therefore, the distance between the two given straight lines is zero.Thus, the given lines intersect each other.

Example 6.36

Find the shortest distance between the two given straight lines $\vec{r}=(2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$ and $\dfrac{x-3}{2}=\dfrac{y}{-1}=\dfrac{z+2}{2}$.

Solution

The parametric form of vector equations of the given straight lines are

$$\vec{r} = (2\hat{i}+3\hat{j}+4\hat{k})+t(-2\hat{i}+\hat{j}-2\hat{k})$$

$$\text{and} \quad \vec{r} = (3\hat{i}-2\hat{k})+t(2\hat{i}-\hat{j}+2\hat{k})$$

Comparing the given two equations with $\vec{r}=\vec{a}+t\vec{b},\ \vec{r}=\vec{c}+s\vec{d}$

we have $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\ \vec{b}=-2\hat{i}+\hat{j}-2\hat{k},\ \vec{c}=3\hat{i}-2\hat{k},\ \vec{d}=2\hat{i}-\hat{j}+2\hat{k}$.

Clearly, $\vec{b}$ is a scalar multiple of $\vec{d}$, and hence the two straight lines are parallel. We know that the shortest distance between two parallel straight lines is given by $d=\dfrac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$.

$$\text{Now,} \quad (\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix} = 12\hat{i}+14\hat{j}-5\hat{k}$$

$$\text{Therefore,} \quad d = \frac{|12\hat{i}+14\hat{j}-5\hat{k}|}{|-2\hat{i}+\hat{j}-2\hat{k}|}=\frac{\sqrt{365}}{3}.$$

■

Example 6.37

Find the coordinate of the foot of the perpendicular drawn from the point $(-1,2,3)$ to the straight line $\vec{r}=(\hat{i}-4\hat{j}+3\hat{k})+t(2\hat{i}+3\hat{j}+\hat{k})$. Also, find the shortest distance from the given point to the straight line.

Solution

Comparing the given equation $\vec{r}=(\hat{i}-4\hat{j}+3\hat{k})+t(2\hat{i}+3\hat{j}+\hat{k})$ with $\vec{r}=\vec{a}+t\vec{b}$, we get $\vec{a}=\hat{i}-4\hat{j}+3\hat{k}$, and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$. We denote the given point $(-1,2,3)$ by D, and the point $(1,-4,3)$ on the straight line by F. If F is the foot of the perpendicular from D to the straight line, then F is of the form $(2t+1,\ 3t-4,t+3)$ and $\overrightarrow{DF}=\overrightarrow{OF}-\overrightarrow{OD}=(2t+2)\hat{i}+(3t-6)\hat{j}+t\hat{k}$.

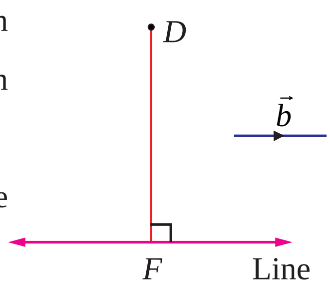


Fig. 6.23

Since $\vec{b}$ is perpendicular to $\overrightarrow{DF}$, we have

$$\vec{b}\cdot\overrightarrow{DF} = 0 \Rightarrow 2(2t+2)+3(3t-6)+1(t)=0 \Rightarrow t=1$$

Therefore, the coordinate of F is $(3,-1,4)$

Now, the perpendicular distance from the given point to the given line is

$$DF = |\overrightarrow{DF}| = \sqrt{4^2+(-3)^2+1^2}=\sqrt{26} \text{ units.}$$

■

EXERCISE 6.5

1. Find the parametric form of vector equation and Cartesian equations of a straight line passing through $(5,2,8)$ and is perpendicular to the straight lines $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+s(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+t(\hat{i}+2\hat{j}+2\hat{k})$.
2. Show that the lines $\vec{r}=(6\hat{i}+\hat{j}+2\hat{k})+s(\hat{i}+2\hat{j}-3\hat{k})$ and $\vec{r}=(3\hat{i}+2\hat{j}-2\hat{k})+t(2\hat{i}+4\hat{j}-5\hat{k})$ are skew lines and hence find the shortest distance between them.
3. If the two lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-m}{2}=z$ intersect at a point, find the value of m.
4. Show that the lines $\frac{x-3}{3}=\frac{y-3}{-1}, z-1=0$ and $\frac{x-6}{2}=\frac{z-1}{3}, y-2=0$ intersect. Also find the point of intersection.
5. Show that the straight lines $x+1=2y=-12z$ and $x=y+2=6z-6$ are skew and hence find the shortest distance between them.
6. Find the parametric form of vector equation of the straight line passing through $(-1,2,1)$ and parallel to the straight line $\vec{r}=(2\hat{i}+3\hat{j}-\hat{k})+t(\hat{i}-2\hat{j}+\hat{k})$ and hence find the shortest distance between the lines.
7. Find the foot of the perpendicular drawn from the point $(5,4,2)$ to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$. Also, find the equation of the perpendicular.

6.8 Different forms of Equation of a plane

We have already seen the notion of a plane.

Definition 6.8

A vector which is perpendicular to a plane is called a **normal** to the plane.

Note

Every normal to a plane is perpendicular to every straight line lying on the plane.
A plane is uniquely fixed if any one of the following is given:

- a unit normal to the plane and the distance of the plane from the origin
- a point of the plane and a normal to the plane
- three non-collinear points of the plane
- a point of the plane and two non-parallel lines or non-parallel vectors which are parallel to the plane
- two distinct points of the plane and a straight line or non-zero vector parallel to the plane but not parallel to the line joining the two points.

Let us find the vector and Cartesian equations of planes using the above situations.

6.8.1 Equation of a plane when a normal to the plane and the distance of the plane from the origin are given

(a) Vector equation of a plane in normal form

Theorem 6.15

The equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\vec{r}\cdot\hat{d}=p$.

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Proof

Consider a plane whose perpendicular distance from the origin is p.

Let A be the foot of the perpendicular from O to the plane.

Let $\hat{d}$ be the unit normal vector in the direction of $\overrightarrow{OA}$.

Then $\overrightarrow{OA} = p\hat{d}$.

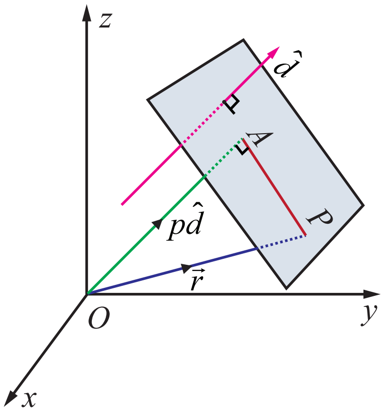

Fig. 6.24

If $\vec{r}$ is the position vector of an arbitrary point P on the plane, then $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{OA}$.

Therefore, $\overrightarrow{AP} \cdot \overrightarrow{OA} = 0 \Rightarrow (\vec{r} - p\hat{d}) \cdot p\hat{d} = 0$

$$\Rightarrow (\vec{r} - p\hat{d}) \cdot \hat{d} = 0$$

which gives $\vec{r} \cdot \hat{d} = p$. ... (1)

The above equation is called the vector equation of the plane in **normal form**. ■

(b) Cartesian equation of a plane in normal form

Let l, m, n be the direction cosines of $\hat{d}$. Then we have $\hat{d} = l\hat{i} + m\hat{j} + n\hat{k}$.

Thus, equation (1) becomes

$$\vec{r} \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p$$

If P is (x,y,z), then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Therefore, $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p$ or $lx + my + nz = p$ ... (2)

Equation (2) is called the Cartesian equation of the plane in **normal form**.

Remark

(i) If the plane passes through the origin, then $p = 0$. So, the equation of the plane is $lx + my + nz = 0$.

(ii) If $\vec{d}$ is normal vector to the plane, then $\hat{d} = \dfrac{\vec{d}}{|\vec{d}|}$ is a unit normal to the plane. So, the vector equation of the plane is $\vec{r} \cdot \dfrac{\vec{d}}{|\vec{d}|} = p$ or $\vec{r} \cdot \vec{d} = q$, where $q = p|\vec{d}|$. The equation $\vec{r} \cdot \vec{d} = q$ is the vector equation of a plane in **standard form.**

Note

In the standard form $\vec{r} \cdot \vec{d} = q$, $\vec{d}$ need not be a unit normal and q need not be the perpendicular distance.

6.8.2 Equation of a plane perpendicular to a vector and passing through a given point

(a) Vector form of equation

Consider a plane passing through a point A with position vector $\vec{a}$ and $\vec{n}$ is a normal vector to the given plane.

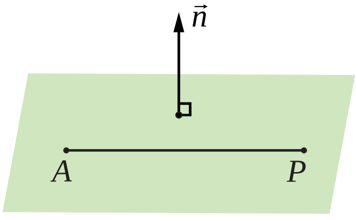

Fig. 6.25

Let $\vec{r}$ be the position vector of an arbitrary point P on the plane.

Then $\overrightarrow{AP}$ is perpendicular to $\vec{n}$.

So, $\overrightarrow{AP} \cdot \vec{n} = 0$ which gives $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$. ... (1)

which is the vector form of the equation of a plane passing through a point with position vector $\vec{a}$ and perpendicular to $\vec{n}$.

Note

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \;\Rightarrow\; \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \;\Rightarrow \vec{r} \cdot \vec{n} = q \text{, where } q = \vec{a} \cdot \vec{n}.$$

(b) Cartesian form of equation

If a, b, c are the direction ratios of $\vec{n}$, then we have $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$. Suppose, A is (x_1, y_1, z_1) then equation (1) becomes $((x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0$. That is,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) \;=\; 0$$

which is the Cartesian equation of a plane, normal to a vector with direction ratios a, b, c and passing through a given point (x_1, y_1, z_1).

6.8.3 Intercept form of the equation of a plane

Let the plane $\vec{r} \cdot \vec{n} = q$ meets the coordinate axes at A, B, C respectively such that the intercepts on the axes are $OA = a,\ OB = b,\ OC = c$. Now position vector of the point A is $a\hat{i}$. Since A lies on the given plane, we have $a\hat{i} \cdot \vec{n} = q$ which gives $\hat{i} \cdot \vec{n} = \frac{q}{a}$. Similarly, since the vectors $b\hat{j}$ and $c\hat{k}$ lie on the given plane, we have $\hat{j} \cdot \vec{n} = \frac{q}{b}$ and $\hat{k} \cdot \vec{n} = \frac{q}{c}$. Substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in $\vec{r} \cdot \vec{n} = q$, we get $x\hat{i} \cdot \vec{n} + y\hat{j} \cdot \vec{n} + z\hat{k} \cdot \vec{n} = q.$ So $x\left(\frac{q}{a}\right) + y\left(\frac{q}{b}\right) + z\left(\frac{q}{c}\right) = q$.

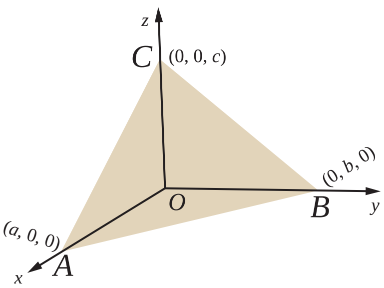

Fig. 6.26

Dividing by q, we get, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. This is called the **intercept form** of equation of the plane having intercepts a, b, c on the x, y, z axes respectively.

Theorem 6.16

The general equation $ax + by + cz + d = 0$ of first degree in x, y, z represents a plane.

Proof

The equation $ax + by + cz + d = 0$ can be written in the vector form as follows

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = -d \quad \text{or} \quad \vec{r} \cdot \vec{n} = -d.$$

Since this is the vector form of the equation of a plane in standard form, the given equation $ax + by + cz + d = 0$ represents a plane. Here $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ is a vector normal to the plane. ■

Note

In the general equation $ax + by + cz + d = 0$ of a plane, a, b, c are direction ratios of the normal to the plane.

Example 6.38

Find the vector and Cartesian form of the equations of a plane which is at a distance of 12 units from the origin and perpendicular to $6\hat{i}+2\hat{j}-3\hat{k}$.

Solution

Let $\vec{d}=6\hat{i}+2\hat{j}-3\hat{k}$ and $p=12$.

If $\hat{d}$ is the unit normal vector in the direction of the vector $6\hat{i}+2\hat{j}-3\hat{k}$,

then $\hat{d}=\dfrac{\vec{d}}{|\vec{d}|}=\dfrac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$.

If $\vec{r}$ is the position vector of an arbitrary point (x, y, z) on the plane, then using $\vec{r}\cdot\hat{d}=p$, the vector equation of the plane in normal form is $\vec{r}\cdot\dfrac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})=12$.

Substituting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in the above equation, we get $(x\hat{i}+y\hat{j}+z\hat{k})\cdot\dfrac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})=12$.

Applying dot product in the above equation and simplifying, we get $6x+2y-3z=84$, which is the Cartesian equation of the required plane.

Example 6.39

If the Cartesian equation of a plane is $3x-4y+3z=-8$, find the vector equation of the plane in the standard form.

Solution

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ is the position vector of an arbitrary point (x, y, z) on the plane, then the given equation can be written as $(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-4\hat{j}+3\hat{k})=-8$ or $(x\hat{i}+y\hat{j}+z\hat{k})\cdot(-3\hat{i}+4\hat{j}-3\hat{k})=8$. That is, $\vec{r}\cdot(-3\hat{i}+4\hat{j}-3\hat{k})=8$ which is the vector equation of the given plane in standard form.

Example 6.40

Find the direction cosines of the normal to the plane and length of the perpendicular from the origin to the plane $\vec{r}\cdot(3\hat{i}-4\hat{j}+12\hat{k})=5$.

Solution

Let $\vec{d}=3\hat{i}-4\hat{j}+12\hat{k}$ and $q=5$.

If $\hat{d}$ is the unit vector in the direction of the vector $3\hat{i}-4\hat{j}+12\hat{k}$, then $\hat{d}=\dfrac{1}{13}(3\hat{i}-4\hat{j}+12\hat{k})$.

Now, dividing the given equation by 13, we get

$$\vec{r}\cdot\left(\frac{3}{13}\hat{i}-\frac{4}{13}\hat{j}+\frac{12}{13}\hat{k}\right)=\frac{5}{13}$$

which is the equation of the plane in the normal form $\vec{r}\cdot\hat{d}=p$.

From this equation, we infer that $\hat{d}=\dfrac{1}{13}(3\hat{i}-4\hat{j}+12\hat{k})$ is a unit vector normal to the plane from the origin. Therefore, the direction cosines of $\hat{d}$ are $\dfrac{3}{13},\dfrac{-4}{13},\dfrac{12}{13}$ and the length of the perpendicular from the origin to the plane is $\dfrac{5}{13}$.

Example 6.41

Find the vector and Cartesian equations of the plane passing through the point with position vector $4\hat{i}+2\hat{j}-3\hat{k}$ and normal to vector $2\hat{i}-\hat{j}+\hat{k}$.

Solution

If the position vector of the given point is $\vec{a}=4\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{n}=2\hat{i}-\hat{j}+\hat{k}$, then the equation of the plane passing through a point and normal to a vector is given by $(\vec{r}-\vec{a})\cdot\vec{n}=0$ or $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$.

Substituting $\vec{a} = 4\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{n}=2\hat{i}-\hat{j}+\hat{k}$ in the above equation, we get

$$\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k}) = (4\hat{i}+2\hat{j}-3\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})$$

Thus, the required vector equation of the plane is $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=3$. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ then we get the Cartesian equation of the plane $2x-y+z=3$. ■

Example 6.42

A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the coordinate axes is a constant. Show that the plane passes through a fixed point

Solution

The equation of the plane having intercepts a,b,c on the x,y,z axes respectively is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Since the sum of the reciprocals of the intercepts on the coordinate axes is a constant, we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=k$, where k is a constant, and which can be written as $\frac{1}{a}\left(\frac{1}{k}\right)+\frac{1}{b}\left(\frac{1}{k}\right)+\frac{1}{c}\left(\frac{1}{k}\right)=1$.

This shows that the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ passes through the fixed point $\left(\frac{1}{k},\frac{1}{k},\frac{1}{k}\right)$. ■

EXERCISE 6.6

1. Find the vector equation of a plane which is at a distance of 7 units from the origin having $3,-4,5$ as direction ratios of a normal to it.
2. Find the direction cosines of the normal to the plane $12x+3y-4z=65$. Also, find the non-parametric form of vector equation of a plane and the length of the perpendicular to the plane from the origin.
3. Find the vector and Cartesian equations of the plane passing through the point with position vector $2\hat{i}+6\hat{j}+3\hat{k}$ and normal to the vector $\hat{i}+3\hat{j}+5\hat{k}$.
4. A plane passes through the point $(-1,1,2)$ and the normal to the plane of magnitude $3\sqrt{3}$ makes equal acute angles with the coordinate axes. Find the equation of the plane.
5. Find the intercepts cut off by the plane $\vec{r}\cdot(6\hat{i}+4\hat{j}-3\hat{k})=12$ on the coordinate axes.
6. If a plane meets the coordinate axes at A,B,C such that the centriod of the triangle ABC is the point (u,v,w), find the equation of the plane.

6.8.4 Equation of a plane passing through three given non-collinear points

(a) Parametric form of vector equation

Theorem 6.17

If three non-collinear points with position vectors $\vec{a}, \vec{b}, \vec{c}$ are given, then the vector equation of the plane passing through the given points in parametric form is

$$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a}), \text{ where } \vec{b} \neq \vec{0}, \vec{c} \neq \vec{0} \text{ and } s, t \in \mathbb{R}.$$

Proof

Consider a plane passing through three non-collinear points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively. Then atleast two of them are non-zero vectors. Let us take $\vec{b} \neq \vec{0}$ and $\vec{c} \neq \vec{0}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the plane. Take a point D on AB (produced) such that $\overrightarrow{AD}$ is parallel to $\overrightarrow{AB}$ and $\overrightarrow{DP}$ is parallel to $\overrightarrow{AC}$. Therefore,

$$\overrightarrow{AD} = s(\vec{b} - \vec{a}), \ \overrightarrow{DP} = t(\vec{c} - \vec{a}).$$

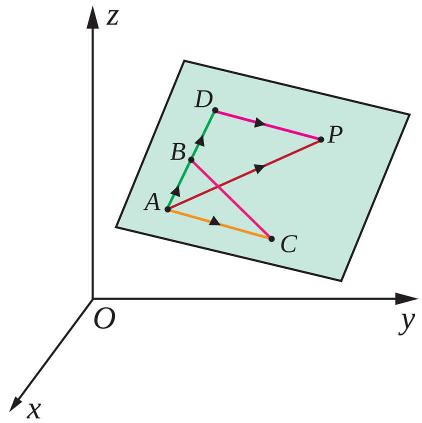

Fig. 6.27

Now, in triangle ADP, we have

$$\overrightarrow{AP} = \overrightarrow{AD} + \overrightarrow{DP} \text{ or } \vec{r} - \vec{a} = s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a}), \text{ where } \vec{b} \neq \vec{0}, \vec{c} \neq \vec{0} \text{ and } s, t \in \mathbb{R}$$

. That is, $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$.

This is the parametric form of vector equation of the plane passing through the given three non-collinear points. ■

(b) Non-parametric form of vector equation

Let $A, B,$ and C be the three non collinear points on the plane with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively. Then atleast two of them are non-zero vectors. Let us take $\vec{b} \neq \vec{0}$ and $\vec{c} \neq \vec{0}$. Now $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$. The vectors $(\vec{b} - \vec{a})$ and $(\vec{c} - \vec{a})$ lie on the plane. Since $\vec{a}, \vec{b}, \vec{c}$ are non-collinear, $\overrightarrow{AB}$ is not parallel to $\overrightarrow{AC}$. Therefore, $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$ is perpendicular to the plane.

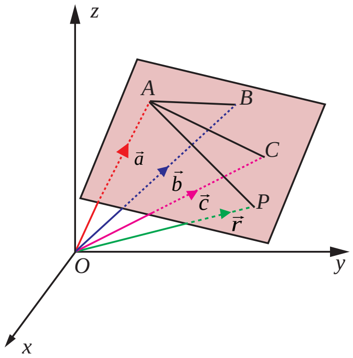

Fig. 6.28

If $\vec{r}$ is the position vector of an arbitrary point $P(x, y, z)$ on the plane, then the equation of the plane passing through the point A with position vector $\vec{a}$ and perpendicular to the vector $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$ is given by

$$(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})) = 0 \quad \text{or} \quad [\vec{r} - \vec{a}, \vec{b} - \vec{a}, \vec{c} - \vec{a}] = 0$$

This is the non-parametric form of vector equation of the plane passing through three non-collinear points.

(c) Cartesian form of equation

If $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) are the coordinates of three non-collinear points A, B, C with position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively and (x, y, z) is the coordinates of the point P with position vector $\vec{r}$, then we have $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, $\vec{c} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Using these vectors, the non-parametric form of vector equation of the plane passing through the given three non-collinear points can be equivalently written as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

which is the Cartesian equation of the plane passing through three non-collinear points.

6.8.5 Equation of a plane passing through a given point and parallel to two given non-parallel vectors.

(a) Parametric form of vector equation

Consider a plane passing through a given point A with position vector $\vec{a}$ and parallel to two given non-parallel vectors $\vec{b}$ and $\vec{c}$. If $\vec{r}$ is the position vector of an arbitrary point P on the plane, then the vectors $(\vec{r}-\vec{a}), \vec{b}$ and $\vec{c}$ are coplanar. So, $(\vec{r}-\vec{a})$ lies in the plane containing $\vec{b}$ and $\vec{c}$. Then, there exists scalars $s, t \in \mathbb{R}$ such that $\vec{r}-\vec{a} = s\vec{b}+t\vec{c}$ which implies

$$\vec{r} = \vec{a}+s\vec{b}+t\vec{c}\text{, where } s, t \in \mathbb{R} \qquad \text{... (1)}$$

This is the parametric form of vector equation of the plane passing through a given point and parallel to two given non-parallel vectors .

(b) Non-parametric form of vector equation

Equation (1) can be equivalently written as

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}) = 0 \qquad \text{... (2)}$$

which is the non-parametric form of vector equation of the plane passing through a given point and parallel to two given non-parallel vectors .

(c) Cartesian form of equation

If $\vec{a} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ and $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then the equation (2) is equivalent to

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

This is the Cartesian equation of the plane passing through a given point and parallel to two given non-parallel vectors.

6.8.6 Equation of a plane passing through two given distinct points and is parallel to a non-zero vector

(a) Parametric form of vector equation

The parametric form of vector equation of the plane passing through two given distinct points A and B with position vectors $\vec{a}$ and $\vec{b}$, and parallel to a non-zero vector $\vec{c}$ is

$$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c} \quad \text{or} \quad \vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c} \qquad \ldots (1)$$

where $s, t \in \mathbb{R}, (\vec{b} - \vec{a})$ and $\vec{c}$ are not parallel vectors.

(b) Non-parametric form of vector equation

Equation (1) can be written equivalently in non-parametric vector form as

$$(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times \vec{c}) = 0 \qquad \ldots (2)$$

where $(\vec{b} - \vec{a})$ and $\vec{c}$ are not parallel vectors.

(c) Cartesian form of equation

If $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k},\ \vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} \neq \vec{0}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then equation (2) is equivalent to

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

This is the required Cartesian equation of the plane.

Example 6.43

Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(0,1,-5)$ and parallel to the straight lines $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + s(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$.

Solution

We observe that the required plane is parallel to the vectors $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}, \vec{c} = \hat{i} + \hat{j} - \hat{k}$ and passing through the point $(0,1,-5)$ with position vector $\vec{a}$. We observe that $\vec{b}$ is not parallel to $\vec{c}$. Then the vector equation of the plane in non-parametric form is given by $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$. …(1)

Substituting $\vec{a} = \hat{j} - 5\hat{k}$ and $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = -9\hat{i} + 8\hat{j} - \hat{k}$ in equation (1), we get

$$(\vec{r} - (\hat{j} - 5\hat{k})) \cdot (-9\hat{i} + 8\hat{j} - \hat{k}) = 0, \text{ which implies that}$$

$$\vec{r} \cdot (-9\hat{i} + 8\hat{j} - \hat{k}) = 13.$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of an arbitrary point on the plane, then from the above equation, we get the Cartesian equation of the plane as $-9x + 8y - z = 13$ or $9x - 8y + z + 13 = 0$. ■

Example 6.44

Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points $(-1, 2, 0)$, $(2, 2 -1)$ and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$.

Solution

The required plane is parallel to the given line and so it is parallel to the vector $\vec{c} = \hat{i} + \hat{j} - \hat{k}$ and the plane passes through the points $\vec{a} = -\hat{i} + 2\hat{j}, \vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$.

- vector equation of the plane in parametric form is $\vec{r} = \vec{a} + s\left(\vec{b} - \vec{a}\right) + t\vec{c}$, where $s,\ t \in \mathbb{R}$ which implies that $\vec{r} = \left(-\hat{i} + 2\hat{j}\right) + s\left(3\hat{i} - \hat{k}\right) + t\left(\hat{i} + \hat{j} - \hat{k}\right)$, where $s, t \in \mathbb{R}$.
- vector equation of the plane in non-parametric form is $(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times \vec{c}) = 0$.

  Now, $(\vec{b} - \vec{a}) \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \hat{i} + 2\hat{j} + 3\hat{k}$,

  we have $(\vec{r} - (-\hat{i} + 2\hat{j})) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0 \Rightarrow \vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 3$
- If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of an arbitrary point on the plane, then from the above equation, we get the Cartesian equation of the plane as $x + 2y + 3z = 3$. ■

EXERCISE 6.7

1. Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point $(2,3,6)$ and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$
2. Find the non-parametric form of vector equation, and Cartesian equations of the plane passing through the points $(2,2,1)$, $(9,3,6)$ and perpendicular to the plane $2x + 6y + 6z = 9$.
3. Find parametric form of vector equation and Cartesian equations of the plane passing through the points $(2,2,1), (1,-2,3)$ and parallel to the straight line passing through the points $(2,1,-3)$ and $(-1,5,-8)$.
4. Find the non-parametric form of vector equation and cartesian equation of the plane passing through the point $(1,-2,4)$ and perpendicular to the plane $x + 2y - 3z = 11$ and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$.
5. Find the parametric form of vector equation, and Cartesian equations of the plane containing the line $\vec{r} = (\hat{i} - \hat{j} + 3\hat{k}) + t(2\hat{i} - \hat{j} + 4\hat{k})$ and perpendicular to plane $\vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 8$.
6. Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the three non-collinear points $(3,6,-2), (-1,-2,6)$, and $(6, 4, -2)$.
7. Find the non-parametric form of vector equation, and Cartesian equations of the plane $\vec{r} = \left(6\hat{i} - \hat{j} + \hat{k}\right) + s\left(-\hat{i} + 2\hat{j} + \hat{k}\right) + t\left(-5\hat{i} - 4\hat{j} - 5\hat{k}\right)$.

6.8.7 Condition for a line to lie in a plane

We observe that a straight line will lie in a plane if every point on the line, lie in the plane and the normal to the plane is perpendicular to the line.

(i) If the line $\vec{r} = \vec{a} + t\vec{b}$ lies in the plane $\vec{r} \cdot \vec{n} = d$, then $\vec{a} \cdot \vec{n} = d$ and $\vec{b} \cdot \vec{n} = 0$.

(ii) If the line $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ lies in the plane $Ax + By + Cz + D = 0$, then $Ax_1 + By_1 + Cz_1 + D = 0$ and $aA + bB + cC = 0$.

Example 6.45

Verify whether the line $\frac{x-3}{-4}=\frac{y-4}{-7}=\frac{z+3}{12}$ lies in the plane $5x-y+z=8$.

Solution

Here, $(x_1,y_1,z_1)=(3,4,-3)$ and direction ratios of the given straight line are $(a,b,c)=(-4,-7,12)$. Direction ratios of the normal to the given plane are $(A,B,C)=(5,-1,1)$.

We observe that, the given point $(x_1,y_1,z_1)=(3,4,-3)$ satisfies the given plane $5x-y+z=8$

Next, $aA+bB+cC=(-4)(5)+(-7)(-1)+(12)(1)=-1\neq 0$. So, the normal to the plane is not perpendicular to the line. Hence, the given line does not lie in the plane.

6.8.8 Condition for coplanarity of two lines

(a) Condition in vector form

The two given non-parallel lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ are coplanar. So they lie in a single plane. Let A and C be the points whose position vectors are $\vec{a}$ and $\vec{c}$. Then A and C lie on the plane. Since $\vec{b}$ and $\vec{d}$ are parallel to the plane, $\vec{b}\times\vec{d}$ is perpendicular to the plane. So $\overrightarrow{AC}$ is perpendicular to $\vec{b}\times\vec{d}$. That is,

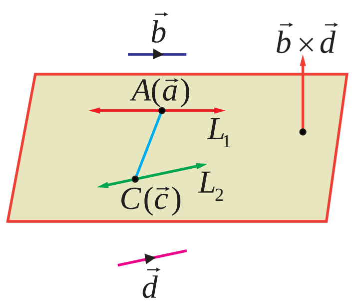

Fig. 6.29

$$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$$

This is the required condition for coplanarity of two lines in vector form.

(b) Condition in Cartesian form

Two lines $\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$$

This is the required condition for coplanarity of two lines in Cartesian form.

6.8.9 Equation of plane containing two non-parallel coplanar lines

(a) Parametric form of vector equation

Let $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ be two non-parallel coplanar lines. Then $\vec{b}\times\vec{d}\neq\vec{0}$. Let P be any point on the plane and let $\vec{r_0}$ be its position vector. Then, the vectors $\vec{r_0}-\vec{a},\vec{b},\vec{d}$ as well as $\vec{r_0}-\vec{c},\vec{b},\vec{d}$ are also coplanar. So, we get $\vec{r_0}-\vec{a}=t\vec{b}+s\vec{d}$ or $\vec{r_0}-\vec{c}=t\vec{b}+s\vec{d}$. Hence, the vector equation in parametric form is $\vec{r}=\vec{a}+t\vec{b}+s\vec{d}$ or $\vec{r}=\vec{c}+t\vec{b}+s\vec{d}$.

(b) Non-parametric form of vector equation

Let $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ be two non-parallel coplanar lines. Then $\vec{b}\times\vec{d}\neq\vec{0}$. Let P be any point on the plane and let $\vec{r_0}$ be its position vector. Then, the vectors $\vec{r_0}-\vec{a},\vec{b},\vec{d}$ as well as $\vec{r_0}-\vec{c},\vec{b},\vec{d}$ are also coplanar. So, we get $\left(\vec{r_0}-\vec{a}\right).\left(\vec{b}\times\vec{d}\right)=0$ or $\left(\vec{r_0}-\vec{c}\right).\left(\vec{b}\times\vec{d}\right)=0$. Hence, the vector equation in non-parametric form is $\left(\vec{r}-\vec{a}\right).\left(\vec{b}\times\vec{d}\right)=0$ or $\left(\vec{r}-\vec{c}\right).\left(\vec{b}\times\vec{d}\right)=0$.

(C) Cartesian form of equation of plane

In Cartesian form the equation of the plane containing the two given coplanar lines

$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$ and $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$ is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \quad \text{or}$$

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Example 6.46

Show that the lines $\vec{r}=\left(-\hat{i}-3\hat{j}-5\hat{k}\right)+s\left(3\hat{i}+5\hat{j}+7\hat{k}\right)$ and $\vec{r}=\left(2\hat{i}+4\hat{j}+6\hat{k}\right)+t\left(\hat{i}+4\hat{j}+7\hat{k}\right)$ are coplanar. Also,find the non-parametric form of vector equation of the plane containing these lines.

Solution

Comparing the two given lines with

$$\vec{r}=\vec{a}+t\vec{b},\ \vec{r}=\vec{c}+s\vec{d}$$

we have, $\vec{a} = -\hat{i}-3\hat{j}-5\hat{k},\ \vec{b}=3\hat{i}+5\hat{j}+7\hat{k},\ \vec{c}=2\hat{i}+4\hat{j}+6\hat{k}$ and $\vec{d}=\hat{i}+4\hat{j}+7\hat{k}$

We know that the two given lines are coplanar , if $(\vec{c}-\vec{a})\cdot\left(\vec{b}\times\vec{d}\right)=0$

Here, $$\vec{b}\times\vec{d}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 7\hat{i}-14\hat{j}+7\hat{k} \text{ and } \vec{c}-\vec{a}=3\hat{i}+7\hat{j}+11\hat{k}$$

Then, $$(\vec{c}-\vec{a})\cdot\left(\vec{b}\times\vec{d}\right) = \left(3\hat{i}+7\hat{j}+11\hat{k}\right)\cdot\left(7\hat{i}-14\hat{j}+7\hat{k}\right)=0.$$

Therefore the two given lines are coplanar.Then we find the non parametric form of vector equation of the plane containing the two given coplanar lines. We know that the plane containing the two given coplanar lines is

$$(\vec{r}-\vec{a})\cdot\left(\vec{b}\times\vec{d}\right) = 0$$

which implies that $\left(\vec{r}-\left(-\hat{i}-3\hat{j}-5\hat{k}\right)\right)\cdot\left(7\hat{i}-14\hat{j}+7\hat{k}\right)=0$. Thus, the required non-parametric vector equation of the plane containing the two given coplanar lines is $\vec{r}\cdot\left(\hat{i}-2\hat{j}+\hat{k}\right)=0$.

EXERCISE 6.8

1. Show that the straight lines $\vec{r} = (5\hat{i}+7\hat{j}-3\hat{k}) + s(4\hat{i}+4\hat{j}-5\hat{k})$ and $\vec{r} = \left(8\hat{i}+4\hat{j}+5\hat{k}\right) + t\left(7\hat{i}+\hat{j}+3\hat{k}\right)$ are coplanar. Find the vector equation of the plane in which they lie.
2. Show that the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{3}$ and $\frac{x-1}{-3} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar. Also, find the plane containing these lines.
3. If the straight lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{m^2}$ and $\frac{x-3}{1} = \frac{y-2}{m^2} = \frac{z-1}{2}$ are coplanar, find the distinct real values of m.
4. If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

6.8.10 Angle between two planes

The angle between two given planes is same as the angle between their normals.

Theorem 6.18

The acute angle θ between the two planes $\vec{r}\cdot\vec{n}_1 = p_1$ and $\vec{r}\cdot\vec{n}_2 = p_2$ is given by

$$\theta = \cos^{-1}\left(\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$$

Proof

If θ is the acute angle between two planes $\vec{r}\cdot\vec{n}_1 = p_1$ and $\vec{r}\cdot\vec{n}_2 = p_2$, then θ is the acute angle between their normal vectors $\vec{n}_1$ and $\vec{n}_2$.

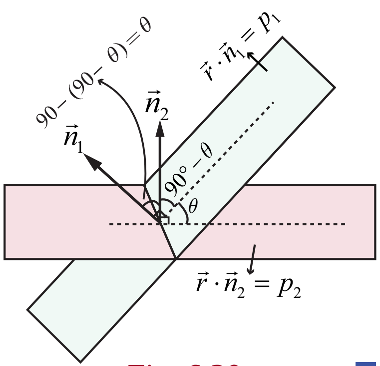

Fig. 6.30

Therefore, $$\cos\theta = \left(\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right) \Rightarrow \theta = \cos^{-1}\left(\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right) \quad \ldots (1)$$

Remark

(i) If two planes $\vec{r}\cdot\vec{n}_1 = p_1$ and $\vec{r}\cdot\vec{n}_2 = p_2$ are perpendicular, then $\vec{n}_1\cdot\vec{n}_2 = 0$

(ii) If the planes $\vec{r}\cdot\vec{n}_1 = p_1$ and $\vec{r}\cdot\vec{n}_2 = p_2$ are parallel, then $\vec{n}_1 = \lambda\vec{n}_2$, where λ is a scalar

(iii) Equation of a plane parallel to the plane $\vec{r}\cdot\vec{n} = p$ is $\vec{r}\cdot\vec{n} = k,\ k\in\mathbb{R}$

Theorem 6.19

The acute angle θ between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ is given by $\theta = \cos^{-1}\left(\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$

Proof

If $\vec{n}_1$ and $\vec{n}_2$ are the vectors normal to the two given planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ respectively. Then, $\vec{n}_1=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{n}_2=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$

Therefore, using equation (1) in theorem 6.18 the acute angle θ between the planes is given by

$$\theta=\cos^{-1}\left(\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

Remark

(i) The planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$

(ii) The planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(iii) Equation of a plane parallel to the plane $ax+by+cz=p$ is $ax+by+cz=k$, $k\in\mathbb{R}$

Example 6.47

Find the acute angle between the planes $\vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k})=11$ and $4x-2y+2z=15$.

Solution

The normal vectors of the two given planes $\vec{r}\cdot(2\hat{i}+2\hat{j}+2\hat{k})=11$ and $4x-2y+2z=15$ are $\vec{n}_1=2\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{n}_2=4\hat{i}-2\hat{j}+2\hat{k}$ respectively.

If θ is the acute angle between the planes, then we have

$$\theta=\cos^{-1}\left(\frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)=\cos^{-1}\left(\frac{\left|(2\hat{i}+2\hat{j}+2\hat{k})\cdot(4\hat{i}-2\hat{j}+2\hat{k})\right|}{|2\hat{i}+2\hat{j}+2\hat{k}||4\hat{i}-2\hat{j}+2\hat{k}|}\right)=\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

6.8.11 Angle between a line and a plane

We know that the angle between a line and a plane is the complement of the angle between the normal to the plane and the line

Let $\vec{r}=\vec{a}+t\vec{b}$ be the equation of the line and $\vec{r}\cdot\vec{n}=p$ be the equation of the plane. We know that $\vec{b}$ is parallel to the given line and $\vec{n}$ is normal to the given plane. If θ is the acute angle between the line and the plane, then the acute angle between $\vec{n}$ and $\vec{b}$ is $\left(\frac{\pi}{2}-\theta\right)$. Therefore,

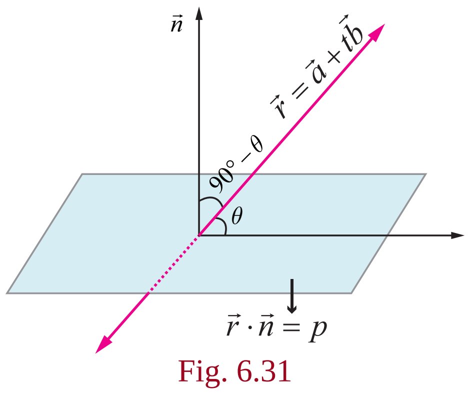

Fig. 6.31

$$\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta=\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}$$

So, the acute angle between the line and the plane is given by $\theta = \sin^{-1}\left(\dfrac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\,|\vec{n}|}\right)$ ... (1)

In Cartesian form if $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ and $ax+by+cz=p$ are the equations of the line and the plane, then $\vec{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$. Therefore, using (1), the acute angle θ between the line and plane is given by

$$\theta = \sin^{-1}\left(\frac{|aa_1 + bb_1 + cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^{\,2}+b_1^{\,2}+c_1^{\,2}}}\right)$$

Remark

(i) If the line is perpendicular to the plane, then the line is parallel to the normal to the plane.

So, $\vec{b}$ is perpendicular to $\vec{n}$. Then we have $\vec{b} = \lambda\vec{n}$ where $\lambda \in \mathbb{R}$,which gives $\dfrac{a_1}{a} = \dfrac{b_1}{b} = \dfrac{c_1}{c}$.

(ii) If the line is parallel to the plane, then the line is perpendicular to the normal to the plane.

Therefore, $\vec{b}\cdot\vec{n} = 0 \Rightarrow aa_1 + bb_1 + cc_1 = 0$

Example 6.48

Find the angle between the straight line $\vec{r} = \left(2\hat{i} + 3\hat{j} + \hat{k}\right) + t\left(\hat{i} - \hat{j} + \hat{k}\right)$ and the plane $2x - y + z = 5$.

Solution

The angle between a line $\vec{r} = \vec{a} + t\vec{b}$ and a plane $\vec{r}\cdot\vec{n} = p$ with normal $\vec{n}$ is $\theta = \sin^{-1}\left(\dfrac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right)$.

Here, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$.

So,we get $\theta = \sin^{-1}\left(\dfrac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\,|\vec{n}|}\right) = \sin^{-1}\left(\dfrac{\left|\left(\hat{i} - \hat{j} + \hat{k}\right)\cdot\left(2\hat{i} - \hat{j} + \hat{k}\right)\right|}{\left|\hat{i} - \hat{j} + \hat{k}\right|\left|2\hat{i} - \hat{j} + \hat{k}\right|}\right) = \sin^{-1}\left(\dfrac{2\sqrt{2}}{3}\right)$ ■

6.8.12 Distance of a point from a plane

(a) Equation of a plane in vector form

Theorem 6.20

The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r}\cdot\vec{n} = p$ is given by

$$\delta = \frac{|\vec{u}\cdot\vec{n} - p|}{|\vec{n}|}.$$

Proof

Let A be the point whose position vector is $\vec{u}$.

Let F be the foot of the perpendicular from the point A to the plane $\vec{r}\cdot\vec{n}=p$. The line joining F and A is parallel to the normal vector $\vec{n}$ and hence its equation is $\vec{r}=\vec{u}+t\vec{n}$.

But F is the point of intersection of the line $\vec{r}=\vec{u}+t\vec{n}$ and the given plane $\vec{r}\cdot\vec{n}=p$. If $\vec{r}_1$ is the position vector of F, then $\vec{r}_1=\vec{u}+t_1\vec{n}$ for some $t_1\in\mathbb{R}$, and $\vec{r}_1\cdot\vec{n}=p$. Eliminating $\vec{r}_1$ we get

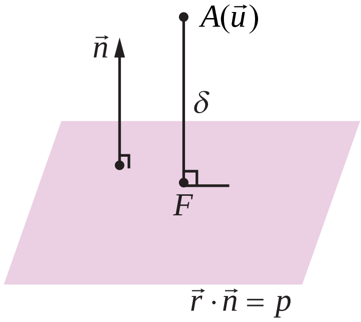

Fig. 6.32

$(\vec{u}+t_1\vec{n})\cdot\vec{n}=p$ which implies $t_1=\dfrac{p-(\vec{u}\cdot\vec{n})}{|\vec{n}|^2}$.

Now, $$\overrightarrow{FA}=\vec{u}-(\vec{u}+t_1\vec{n}) \;=\; -t_1\vec{n}=\left(\frac{(\vec{u}\cdot\vec{n})-p}{|\vec{n}|^2}\right)\vec{n}$$

Therefore, the length of the perpendicular from the point A to the given plane is

$$\delta \;=\; \left|\overrightarrow{FA}\right|=\left|\left(\frac{(\vec{u}\cdot\vec{n})-p}{|\vec{n}|^2}\right)\vec{n}\right|=\left|\frac{(\vec{u}\cdot\vec{n})-p}{|\vec{n}|}\right|$$

The position vector of the foot F of the perpendicular AF is given by

$$\vec{r}_1 \;=\; \vec{u}+t_1\vec{n} \quad \text{or}$$

$$\vec{r}_1 \;=\; \vec{u}+\left(\frac{p-\vec{u}\cdot\vec{n}}{|\vec{n}|^2}\right)\vec{n}$$

(b) Equation of a plane in Cartesian form

In Caretesian form if $A(x_1,y_1,z_1)$ is the given point with position vector $\vec{u}$ and $ax+by+cz=p$ is the Cartesian equation of the given plane, then $\vec{u}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$. Therefore, using these vectors in $\delta=\dfrac{|\vec{u}\cdot\vec{n}-p|}{|\vec{n}|}$, we get the perpendicular distance from a point to the plane in Cartesian form as

$$\delta=\left|\frac{ax_1+by_1+cz_1-p}{\sqrt{a^2+b^2+c^2}}\right|=\frac{|ax_1+by_1+cz_1-p|}{\sqrt{a^2+b^2+c^2}}$$

Remark

The perpendicular distance from the origin to the plane $ax+by+cz+d=0$ is given by

$$\delta=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

Example 6.49

Find the distance of a point $(2,5,-3)$ from the plane $\vec{r}\cdot(6\hat{i}-3\hat{j}+2\hat{k})=5$.

Solution

Comparing the given equation of the plane with $\vec{r}\cdot\vec{n}=p$, we have $\vec{n}=6\hat{i}-3\hat{j}+2\hat{k}$.

We know that the perpendicular distance from the given point with position vector $\vec{u}$ to the plane $\vec{r}\cdot\vec{n}=p$ is given by $\delta=\frac{|\vec{u}\cdot\vec{n}-p|}{|\vec{n}|}$. Therefore, substituting $\vec{u}=(2,5,-3)=2\hat{i}+5\hat{j}-3\hat{k}$ and $\vec{n}=6\hat{i}-3\hat{j}+2\hat{k}$ in the formula, we get

$$\delta=\frac{|\vec{u}\cdot\vec{n}-p|}{|\vec{n}|}=\frac{\left|\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\cdot\left(6\hat{i}-3\hat{j}+2\hat{k}\right)-5\right|}{\left|6\hat{i}-3\hat{j}+2\hat{k}\right|}=2\text{ units.}$$

Example 6.50

Find the distance of the point $(5,-5,-10)$ from the point of intersection of a straight line passing through the points $A(4,1,2)$ and $B(7,5,4)$ with the plane $x-y+z=5$.

Solution

The Cartesian equation of the straight line joining A and B is

$$\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}=t\quad\text{(say)}.$$

Therefore, an arbitrary point on the straight line is of the form $(3t+4,4t+1,2t+2)$. To find the point of intersection of the straight line and the plane, we substitute $x=3t+4,\ y=4t+1,\ z=2t+2$ in $x-y+z=5$, and we get $t=0$. Therefore,the point of intersection of the straight line is (4, 1, 2). Now, the distance between the two points (4, 1, 2) and $(5,-5,-10)$ is

$$\sqrt{(4-5)^2+(1+5)^2+(2+10)^2}=\sqrt{181}\text{ units.}$$

6.8.13 Distance between two parallel planes

Theorem 6.21

The distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is given by $\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.

Proof

Let $A(x_1,y_1,z_1)$ be any point on the plane $ax+by+cz+d_2=0$, then we have

$ax_1+by_1+cz_1+d_2=0\Rightarrow\ ax_1+by_1+cz_1=-d_2$

The distance of the plane $ax+by+cz+d_1=0$ from the point $A(x_1,y_1,z_1)$ is given by

$$\delta=\frac{|ax_1+by_1+cz_1+d_1|}{\sqrt{a^2+b^2+c^2}}=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

Hence, the distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is given by $\delta=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$.

Example 6.51

Find the distance between the parallel planes $x+2y-2z+1=0$ and $2x+4y-4z+5=0$.

Solution

We know that the formula for the distance between two parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is $\delta=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$. Rewrite the second equation as $x+2y-2z+\frac{5}{2}=0$.

Comparing the given equations with the general equations, we get $a=1, b=2, c=-2, d_1=1, d_2=\frac{5}{2}$.

Substituting these values in the formula, we get the distance

$$\delta=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}=\frac{\left|1-\frac{5}{2}\right|}{\sqrt{1^2+2^2+\left(-2^2\right)}}=\frac{1}{2} \text{ units.}$$

Example 6.52

Find the distance between the planes $\vec{r}\cdot\left(2\hat{i}-\hat{j}-2\hat{k}\right)=6$ and $\vec{r}\cdot\left(6\hat{i}-3\hat{j}-6\hat{k}\right)=27$

Solution

Let $\vec{u}$ be the position vector of an arbitrary point on the plane $\vec{r}\cdot(2\hat{i}-\hat{j}-2\hat{k})=6$. Then, we have

$$\vec{u}\cdot(2\hat{i}-\hat{j}-2\hat{k})=6. \qquad \text{... (1)}$$

If δ is the distance between the given planes, then δ is the perpendicular distance from $\vec{u}$ to the plane

$$\vec{r}\cdot(6\hat{i}-3\hat{j}-6\hat{i})=27.$$

Therefore, $\delta=\frac{|\vec{u}\cdot\vec{n}-p|}{|\vec{n}|}=\left|\frac{\vec{u}\cdot(6\hat{i}-3\hat{j}-6\hat{k})-27}{\sqrt{6^2+(-3)^2+(-6)^2}}\right|=\left|\frac{3(\vec{u}\cdot(2\hat{i}-\hat{j}-2\hat{k}))-27}{9}\right|=\left|\frac{(3(6)-27}{9}\right|=1$ unit.

6.8.14 Equation of line of intersection of two planes

Let $\vec{r}\cdot\vec{n}=p$ and $\vec{r}\cdot\vec{m}=q$ be two non-parallel planes. We know that $\vec{n}$ and $\vec{m}$ are perpendicular to the given planes respectively. So, the line of intersection of these planes is perpendicular to both $\vec{n}$ and $\vec{m}$. Therefore, it is parallel to the vector $\vec{n}\times\vec{m}$. Let $\vec{n}\times\vec{m}=l_1\hat{i}+l_2\hat{j}+l_3\hat{k}$

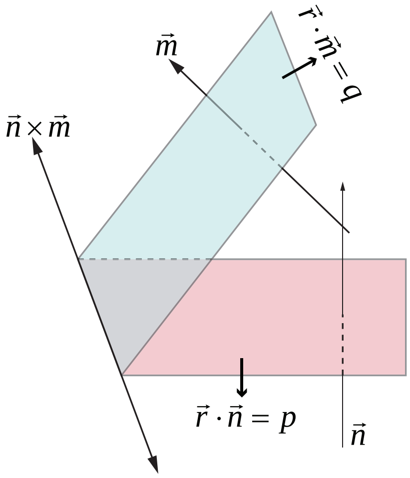

Fig. 6.33

Consider the equations of two planes $a_1x+b_1y+c_1z=p$ and $a_2x+b_2y+c_2z=q$. The line of intersection of the two given planes intersects atleast one of the coordinate planes. For simplicity, we assume that the line meets the coordinate plane $z=0$. Substitute $z=0$ and obtain the two equations $a_1x+b_1y-p=0$ and $a_2x+b_2y-q=0$. Then by solving these equations, we get the values of x and y as x_1 and y_1 respectively.

So, $(x_1, y_1, 0)$ is a point on the required line, which is parallel to $l_1\hat{i} + l_2\hat{j} + l_3\hat{k}$. So, the equation of the line is $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{l_2} = \dfrac{z - 0}{l_3}$.

6.8.15 Equation of a plane passing through the line of intersection of two given planes

Theorem 6.22

The vector equation of a plane which passes through the line of intersection of the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by $(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$, where $\lambda \in \mathbb{R}$.

Proof

Consider the equation

$(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$ ... (1)

The above equation can be simplified as

$\vec{r} \cdot (\vec{n}_1 + \lambda\vec{n}_2) - (d_1 + \lambda d_2) = 0$ ... (2)

Put $\vec{n} = \vec{n}_1 + \lambda\vec{n}_2$, $d = (d_1 + \lambda d_2)$.

Then the equation (2) becomes

$\vec{r} \cdot \vec{n} = d$ ... (3)

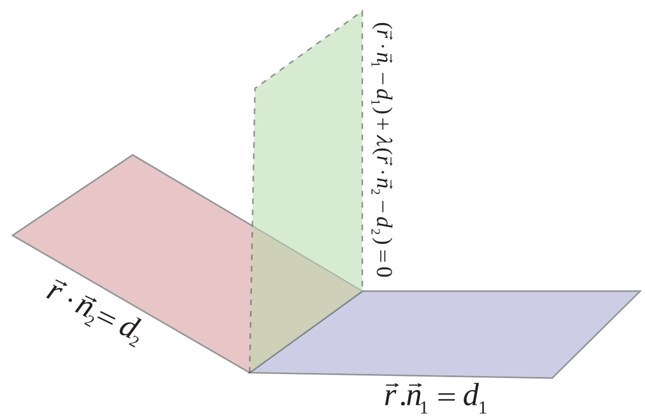

Fig. 6.34

The equation (3) represents a plane. Hence (1) represents a plane.

Let $\vec{r}_1$ be the position vector of any point on the line of intersection of the plane. Then $\vec{r}_1$ satisfies both the equations $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$. So, we have

$$\vec{r}_1 \cdot \vec{n}_1 = d_1 \quad \text{... (4)}$$

$$\text{and} \quad \vec{r}_2 \cdot \vec{n}_2 = d_2 \quad \text{... (5)}$$

By (4) and (5), $\vec{r}_1$ satisfies (1). So, any point on the line of intersection lies on the plane (1). This proves that the plane (1) passes through the line of intersection.

The **cartesian equation** of a plane which passes through the line of intersection of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by

$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$ ■

Example 6.53

Find the equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) + 1 = 0$ and $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) = 2$ and the point $(-1, 2, 1)$.

Solution

We know that the vector equation of a plane passing through the line of intersection of the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by $(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$

Substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{n}_1 = \hat{i} + \hat{j} + \hat{k}$, $\vec{n}_2 = 2\hat{i} - 3\hat{j} + 5\hat{k}$, $d_1 = 1$, $d_2 = -2$ in the above equation, we get

$(x + y + z + 1) + \lambda(2x - 3y + 5z - 2) = 0$

Since this plane passes through the point $(-1,2,1)$, we get $\lambda=\frac{3}{5}$, and hence the required equation of the plane is $11x-4y+20z=1$.

Example 6.54

Find the equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ and is perpendicular to the plane $x+y-3z-5=0$.

Solution

The equation of the plane passing through the intersection of the planes $2x+3y-z+7=0$ and $x+y-2z+5=0$ is $(2x+3y-z+7)+\lambda(x+y-2z+5)=0$ or

$$(2+\lambda)x+(3+\lambda)y+(-1-2\lambda)z+(7+5\lambda)=0$$

since this plane is perpendicular to the given plane $x+y-3z-5=0$, the normals of these two planes are perpendicular to each other. Therefore, we have

$$(1)(2+\lambda)+(1)(3+\lambda)+(-3)(-1-2\lambda)z=0$$

which implies that $\lambda=-1$. Thus the required equation of the plane is

$$(2x+3y-z+7)-(x+y-2z+5)=0 \Rightarrow x+2y+z+2=0 .$$

6.9 Image of a Point in a Plane

Let A be the given point whose position vector is $\vec{u}$. Let $\vec{r}\cdot\vec{n}=p$ be the equation of the plane. Let $\vec{v}$ be the position vector of the mirror image A' of A in the plane. Then $\overrightarrow{AA'}$ is perpendicular to the plane. So it is parallel to $\vec{n}$. Then

$$\overrightarrow{AA'}=\lambda\vec{n} \text{ or } \vec{v}-\vec{u}=\lambda\vec{n} \Rightarrow \vec{v}=\vec{u}+\lambda\vec{n} \quad \text{... (1)}$$

Let M be the middle point of AA'. Then the position vector of M is $\frac{\vec{u}+\vec{v}}{2}$. But M lies on the plane.

So, $\left(\frac{\vec{u}+\vec{v}}{2}\right)\cdot\vec{n}=p$. ... (2)

Sustituting (1) in (2), we get

$$\left(\frac{\vec{u}+\lambda\vec{n}+\vec{u}}{2}\right)\cdot\vec{n}=p \Rightarrow \lambda=\frac{2\left[p-(\vec{u}\cdot\vec{n})\right]}{|\vec{n}|^2}$$

Therefore, the position vector of A'

is $\vec{v}=\vec{u}+\frac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}$

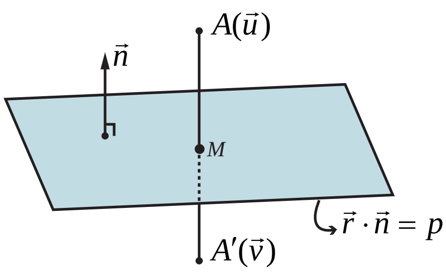

Fig. 6.35

Note

The mid point of M of AA' is the foot of the perpendicular from the point A to the plane $\vec{r}\cdot\vec{n}=p$. So the position vector of the foot M of the perpendicular is given by .

$$\frac{\vec{u}+\vec{v}}{2}=\frac{\vec{u}}{2}+\frac{1}{2}\left(\vec{u}+\frac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}\vec{n}\right)$$

6.9.1 The coordinates of the image of a point in a plane

Let (a_1, a_2, a_3) be the point $\vec{u}$ whose image in the plane is required. Then $\vec{u} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$.

Let $ax + by + cz = d$ be the equation of the given plane.Writing the equation in the vector form we get $\vec{r}\cdot\vec{n} = p$ where $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ .Then the position vector of the image is

$$\vec{v} = \vec{u} + \frac{2\left[p - (\vec{u}\cdot\vec{n})\right]}{|\vec{n}|^2}\vec{n}.$$

If $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, then $v_1 = a_1 + 2a\alpha$, $v_2 = a_2 + 2a\alpha$, $v_3 = a_3 + 2a\alpha$

$$\text{where } \alpha = \frac{2\left[p - (aa_1 + ba_2 + ca_3)\right]}{a^2 + b^2 + c^2}.$$

Example 6.55

Find the image of the point whose position vector is $\hat{i} + 2\hat{j} + 3\hat{k}$ in the plane $\vec{r}\cdot\left(\hat{i} + 2\hat{j} + 4\hat{k}\right) = 38$.

Solution

Here, $\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{n} = \hat{i} + 2\hat{j} + 4\hat{k}$, $p = 38$. Then the position vector of the image $\vec{v}$ of

$\vec{u} = \hat{i} + 2\hat{j} + 3\hat{k}$ is given by $\vec{v} = \vec{u} + \dfrac{2\left[p - (\vec{u}\cdot\vec{n})\right]}{|\vec{n}|^2}\vec{n}$.

$$\vec{v} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \frac{2\left[38 - \left(\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)\cdot\left(\hat{i} + 2\hat{j} + 4\hat{k}\right)\right)\right]}{\left(\hat{i} + 2\hat{j} + 4\hat{k}\right)\cdot\left(\hat{i} + 2\hat{j} + 4\hat{k}\right)}\left(\hat{i} + 2\hat{j} + 4\hat{k}\right).$$

That is, $\vec{v} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + 2\left(\dfrac{38 - 17}{21}\right)\left(\hat{i} + 2\hat{j} + 4\hat{k}\right) = 3\hat{i} + 6\hat{j} + 11\hat{k}$.

Therefore, the image of the point with position vector $\hat{i} + 2\hat{j} + 3\hat{k}$ is $3\hat{i} + 6\hat{j} + 11\hat{k}$. ■

Note

The foot of the perpendicular from the point with position vector $\hat{i} + 2\hat{j} + 3\hat{k}$ in the given plane is

$$\frac{(\hat{i} + 2\hat{j} + 3\hat{k}) + (3\hat{i} + 6\hat{j} + 11\hat{k})}{2} = 2\hat{i} + 4\hat{j} + 7\hat{k}.$$

6.10 Meeting Point of a Line and a Plane

Theorem 6.23

The position vector of the point of intersection of the straight line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = p$ is $\vec{a} + \left(\dfrac{p - (\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b}$, provided $\vec{b}\cdot\vec{n} \neq 0$.

Proof

Let $\vec{r} = \vec{a} + t\vec{b}$ be the equation of the given line which is not parallel to the given plane whose equation is $\vec{r}\cdot\vec{n} = p$. So, $\vec{b}\cdot\vec{n} \neq 0$.

Let $\vec{u}$ be the position vector of the meeting point of the line with the plane. Then $\vec{u}$ satisfies both $\vec{r} = \vec{a} + t\vec{b}$ and $\vec{r}\cdot\vec{n} = p$ for some value of t, say t_1. So, We get

$$\vec{u} = \vec{a} + t\vec{b} \quad \text{... (1)}$$

$$\vec{u}\cdot\vec{n} = p \quad \text{...(2)}$$

Sustituting (1) in (2), we get

$$\left(\vec{a} + t_1\vec{b}\right)\cdot\vec{n} = p$$

or $\quad \vec{a}\cdot\vec{n} + t_1\left(\vec{b}\cdot\vec{n}\right) = p$

or $\quad t_1 = \dfrac{p - (\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}} \quad \text{...(3)}$

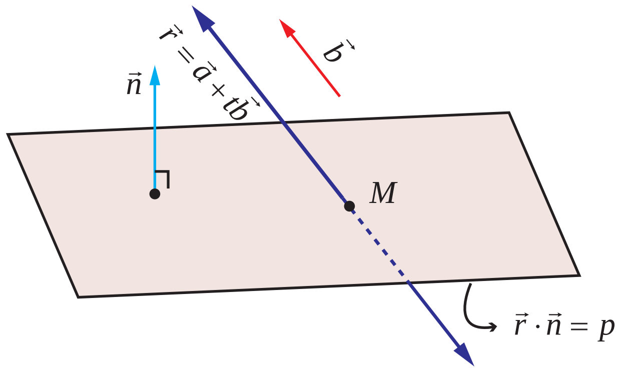

Fig. 6.36

Sustituting (3) in (1), we get

$$\vec{u} = \vec{a} + \left(\frac{p - (\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b},\ \vec{b}\cdot\vec{n} \neq 0$$

Example 6.56

Find the coordinates of the point where the straight line $\vec{r} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right) + t\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ intersects the plane $x - y + z - 5 = 0$.

Solution

Here, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k},\ \vec{b} = 3\hat{i} + 4\hat{j} + 2\hat{k}$.

The vector form of the given plane is $\vec{r}\cdot\left(\hat{i} - \hat{j} + \hat{k}\right) = 5$. Then $\vec{n} = \hat{i} - \hat{j} + \hat{k}$ and $p = 5$.

We know that the position vector of the point of intersection of the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}\cdot\vec{d} = p$ is given by $\vec{u} = \vec{a} + \left(\dfrac{p - (\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b}$, where $\vec{b}\cdot\vec{n} \neq 0$.

Clearly, we observe that $\vec{b}\cdot\vec{n} \neq 0$.

Now, $\dfrac{p - \vec{a}\cdot\vec{n}}{\vec{b}\cdot\vec{n}} = \dfrac{5 - \left(2\hat{i} - \hat{j} + 2\hat{k}\right)\cdot\left(\hat{i} - \hat{j} + \hat{k}\right)}{\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)\cdot\left(\hat{i} - \hat{j} + \hat{k}\right)} = 0$. Therefore,the position vector of the point of intersection of the given line and the given plane is

$$\vec{r} = \left(2\hat{i} - \hat{j} + 2\hat{k}\right) + (0)\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right) = 2\hat{i} - \hat{j} + 2\hat{k}$$

That is, the given straight line intersects the plane at the point $(2, -1, 2)$.

Aliter

The Cartesian equation of the given straight line is $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{2} = t$ (say)

We know that any point on the given straight line is of the form $(3t + 2, 4t - 1, 2t + 2)$. If the given line and the plane intersects, then this point lies on the given pane $x - y + z - 5 = 0$.

So, $(3t + 2) - (4t - 1) + (2t + 2) - 5 = 0 \Rightarrow t = 0$.

Therefore, the given line intersects the given plane at the point $(2, -1, 2)$

EXERCISE 6.9

1. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}\cdot\left(2\hat{i}-7\hat{j}+4\hat{k}\right)=3$ and $3x-5y+4z+11=0$, and the point $(-2,1,3)$.
2. Find the equation of the plane passing through the line of intersection of the planes $x+2y+3z=2$ and $x-y+z=3$, and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3,1,-1)$.
3. Find the angle between the line $\vec{r}=\left(2\hat{i}-\hat{j}+\hat{k}\right)+t\left(\hat{i}+2\hat{j}-2\hat{k}\right)$ and the plane $\vec{r}\cdot\left(6\hat{i}+3\hat{j}+2\hat{k}\right)=8$
4. Find the angle between the planes $\vec{r}\cdot\left(\hat{i}+\hat{j}-2\hat{k}\right)=3$ and $2x-2y+z=2$.
5. Find the equation of the plane which passes through the point $(3,4,-1)$ and is parallel to the plane $2x-3y+5z+7=0$. Also, find the distance between the two planes.
6. Find the length of the perpendicular from the point $(1,-2,3)$ to the plane $x-y+z=5$.
7. Find the point of intersection of the line $x-1=\frac{y}{2}=z+1$ with the plane $2x-y+2z=2$. Also, find the angle between the line and the plane.
8. Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point (4,3,2) to the plane $x+2y+3z=2$.

EXERCISE 6.10

G58SL

Choose the correct or the most suitable answer from the given four alternatives :

1. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a},\vec{c},\vec{b}]$ is equal to

   (1) 2 (2) -1 (3) 1 (4) 0

2. If a vector $\vec{\alpha}$ lies in the plane of $\vec{\beta}$ and $\vec{\gamma}$, then

   (1) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=1$ (2) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=-1$ (3) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=0$ (4) $[\vec{\alpha},\vec{\beta},\vec{\gamma}]=2$

3. If $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$, then the value of $[\vec{a},\vec{b},\vec{c}]$ is

   (1) $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ (2) $\frac{1}{3}|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ (3) 1 (4) -1

4. If $\vec{a},\vec{b},\vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$, and is parallel to $\vec{c}$ then $\vec{a}\times(\vec{b}\times\vec{c})$ is equal to

   (1) $\vec{a}$ (2) $\vec{b}$ (3) $\vec{c}$ (4) $\vec{0}$

5. If $[\vec{a},\vec{b},\vec{c}]=1$, then the value of $\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a})\cdot\vec{b}}+\frac{\vec{b}\cdot(\vec{c}\times\vec{a})}{(\vec{a}\times\vec{b})\cdot\vec{c}}+\frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{(\vec{c}\times\vec{b})\cdot\vec{a}}$ is

   (1) 1 (2) -1 (3) 2 (4) 3

6. The volume of the parallelepiped with its edges represented by the vectors $\hat{i}+\hat{j},\ \hat{i}+2\hat{j},\ \hat{i}+\hat{j}+\pi\hat{k}$ is

   (1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) π (4) $\frac{\pi}{4}$

7. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $[\vec{a}, \vec{b}, \vec{a}\times\vec{b}] = \frac{1}{4}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

8. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+\hat{j}$, $\vec{c} = \hat{i}$ and $(\vec{a}\times\vec{b})\times\vec{c} = \lambda\vec{a}+\mu\vec{b}$, then the value of $\lambda+\mu$ is

(1) 0 (2) 1 (3) 6 (4) 3

9. If $\vec{a},\vec{b},\vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a}, \vec{b}, \vec{c}] = 3$, then $\{[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]\}^2$ is equal to

(1) 81 (2) 9 (3) 27 (4)18

10. If $\vec{a},\vec{b},\vec{c}$ are three non-coplanar unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is

(1) $\frac{\pi}{2}$ (2) $\frac{3\pi}{4}$ (3) $\frac{\pi}{4}$ (4) π

11. If the volume of the parallelepiped with $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$ as coterminous edges is 8 cubic units, then the volume of the parallelepiped with $(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c}), (\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})$ and $(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})$ as coterminous edges is,

(1) 8 cubic units (2) 512 cubic units (3) 64 cubic units (4) 24 cubic units

12. Consider the vectors $\vec{a},\vec{b},\vec{c},\vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$. Let P_1 and P_2 be the planes determined by the pairs of vectors $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$ respectively. Then the angle between P_1 and P_2 is

(1) 0° (2) 45° (3) 60° (4) 90°

13. If $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\times\vec{c}$, where $\vec{a},\vec{b},\vec{c}$ are any three vectors such that $\vec{b}\cdot\vec{c} \neq 0$ and $\vec{a}\cdot\vec{b} \neq 0$, then $\vec{a}$ and $\vec{c}$ are

(1) perpendicular (2) parallel

(3) inclined at an angle $\frac{\pi}{3}$ (4) inclined at an angle $\frac{\pi}{6}$

14. If $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-5\hat{k}$, $\vec{c} = 3\hat{i}+5\hat{j}-\hat{k}$, then a vector perpendicular to $\vec{a}$ and lies in the plane containing $\vec{b}$ and $\vec{c}$ is

(1) $-17\hat{i}+21\hat{j}-97\hat{k}$ (2) $17\hat{i}+21\hat{j}-123\hat{k}$

(3) $-17\hat{i}-21\hat{j}+97\hat{k}$ (4) $-17\hat{i}-21\hat{j}-97\hat{k}$

15. The angle between the lines $\frac{x-2}{3} = \frac{y+1}{-2}, z = 2$ and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{\pi}{2}$

16. If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x+3y-\alpha z+\beta=0$, then (α,β) is

(1) $(-5,5)$ (2) $(-6,7)$ (3) $(5,-5)$ (4) $(6,-7)$

17. The angle between the line $\vec{r}=(\hat{i}+2\hat{j}-3\hat{k})+t(2\hat{i}+\hat{j}-2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+\hat{j})+4=0$ is

(1) $0°$ (2) $30°$ (3) $45°$ (4) $90°$

18. The coordinates of the point where the line $\vec{r}=(6\hat{i}-\hat{j}-3\hat{k})+t(-\hat{i}+4\hat{k})$ meets the plane $\vec{r}.(\hat{i}+\hat{j}-\hat{k})=3$ are

(1) $(2,1,0)$ (2) $(7,-1,-7)$ (3) $(1,2,-6)$ (4) $(5,-1,1)$

19. Distance from the origin to the plane $3x-6y+2z+7=0$ is

(1) 0 (2) 1 (3) 2 (4) 3

20. The distance between the planes $x+2y+3z+7=0$ and $2x+4y+6z+7=0$ is

(1) $\frac{\sqrt{7}}{2\sqrt{2}}$ (2) $\frac{7}{2}$ (3) $\frac{\sqrt{7}}{2}$ (4) $\frac{7}{2\sqrt{2}}$

21. If the direction cosines of a line are $\frac{1}{c},\frac{1}{c},\frac{1}{c}$, then

(1) $c=\pm 3$ (2) $c=\pm\sqrt{3}$ (3) $c>0$ (4) $0<c<1$

22. The vector equation $\vec{r}=(\hat{i}-2\hat{j}-\hat{k})+t(6\hat{j}-\hat{k})$ represents a straight line passing through the points

(1) $(0,6,-1)$ and $(1,-2,-1)$ (2) $(0,6,-1)$ and $(-1,-4,-2)$

(3) $(1,-2,-1)$ and $(1,4,-2)$ (4) $(1,-2,-1)$ and $(0,-6,1)$

23. If the distance of the point $(1,1,1)$ from the origin is half of its distance from the plane $x+y+z+k=0$, then the values of k are

(1) ± 3 (2) ± 6 (3) $-3,9$ (4) $3,-9$

24. If the planes $\vec{r}.(2\hat{i}-\lambda\hat{j}+\hat{k})=3$ and $\vec{r}.(4\hat{i}+\hat{j}-\mu\hat{k})=5$ are parallel, then the value of λ and μ are

(1) $\frac{1}{2},-2$ (2) $-\frac{1}{2},2$ (3) $-\frac{1}{2},-2$ (4) $\frac{1}{2},2$

25. If the length of the perpendicular from the origin to the plane $2x+3y+\lambda z=1$, $\lambda>0$ is $\frac{1}{5}$, then the value of λ is

(1) $2\sqrt{3}$ (2) $3\sqrt{2}$ (3) 0 (4) 1

SUMMARY

1. For a given set of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, the scalar $(\vec{a}\times\vec{b})\cdot\vec{c}$ is called a **scalar triple product** of $\vec{a}, \vec{b}, \vec{c}$.
2. The **volume of the parallelepiped** formed by using the three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ as co-terminus edges is given by $\left|(\vec{a}\times\vec{b})\cdot\vec{c}\right|$.
3. The scalar triple product of three non-zero vectors is zero if and only if the three vectors are **coplanar**.
4. Three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar, if, and only if there exist scalars $r,s,t\in\mathbb{R}$ such that atleast one of them is non-zero and $r\vec{a}+s\vec{b}+t\vec{c}=\vec{0}$.
5. If $\vec{a},\vec{b},\vec{c}$ and $\vec{p},\vec{q},\vec{r}$ are any two systems of three vectors, and if $\vec{p}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}$, $\vec{q}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$, and, $\vec{r}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$, then $\left[\vec{p},\vec{q},\vec{r}\right]=\begin{vmatrix} x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}\left[\vec{a},\vec{b},\vec{c}\right]$.
6. For a given set of three vectors $\vec{a}, \vec{b}, \vec{c}$, the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is called **vector triple product**.
7. For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$.
8. Parametric form of the vector equation of a straight line that passes through a given point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r}=\vec{a}+t\vec{b}$, where $t\in\mathbb{R}$.
9. Cartesian equations of a straight line that passes through the point (x_1, y_1, z_1) and parallel to a vector with direction ratios b_1, b_2, b_3 are $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$.
10. Any point on the line $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$ is of the form $(x_1+tb_1,\ y_1+tb_2,\ z_1+tb_3)$, $t\in\mathbb{R}$.
11. Parametric form of vector equation of a straight line that passes through two given points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+t\left(\vec{b}-\vec{a}\right)$, $t\in\mathbb{R}$.
12. Cartesian equations of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$.
13. If θ is the acute angle between two straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$, then $\theta=\cos^{-1}\left(\dfrac{\left|\vec{b}\cdot\vec{d}\right|}{\left|\vec{b}\right|\left|\vec{d}\right|}\right)$
14. Two lines are said to be coplanar if they lie in the same plane.
15. Two lines in space are called **skew lines** if they are not parallel and do not intersect
16. The shortest distance between the two skew lines is the length of the line segment perpendicular to both the skew lines.
17. The shortest distance between the two skew lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is $\delta=\dfrac{\left|(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})\right|}{\left|\vec{b}\times\vec{d}\right|}$, where $|\vec{b}\times\vec{d}|\neq 0$.

18. Two straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ intersect each other if $(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=0$

19. The shortest distance between the two parallel lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{b}$ is $d=\dfrac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$, where $|\vec{b}|\neq 0$

20. If two lines $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$ and $\dfrac{x-x_2}{d_1}=\dfrac{y-y_2}{d_2}=\dfrac{z-z_2}{d_3}$ intersect, then
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$$

21. A straight line which is perpendicular to a plane is called a normal to the plane.
22. The equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$ is $\vec{r}\cdot\hat{d}=p$ (normal form)
23. Cartesian equation of the plane in normal form is $lx+my+nz=p$
24. Vector form of the equation of a plane passing through a point with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n}=0$.
25. Cartesian equation of a plane normal to a vector with direction ratios a,b,c and passing through a given point (x_1,y_1,z_1) is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.
26. Intercept form of the equation of the plane $\vec{r}\cdot\vec{n}=q$, having intercepts a,b,c on the x,y,z axes respectively is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$.
27. Parametric form of vector equation of the plane passing through three given non-collinear points is $\vec{r}=\vec{a}+s(\vec{b}-\vec{a})+t(\vec{c}-\vec{a})$
28. Cartesian equation of the plane passing through three non-collinear points is
$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0.$$
29. A straight will lie on a plane if every point on the line, lie in the plane and the normal to the plane is perpendicular to the line.
30. The two given non-parallel lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ are coplanar if $(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=0$.
31. Two lines $\dfrac{x-x_1}{b_1}=\dfrac{y-y_1}{b_2}=\dfrac{z-z_1}{b_3}$ and $\dfrac{x-x_2}{d_1}=\dfrac{y-y_2}{d_2}=\dfrac{z-z_2}{d_3}$ are coplanar if
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}=0$$
32. Non-parametric form of vector equation of the plane containing the two coplanar lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is $(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{d})=0$ or $(\vec{r}-\vec{c})\cdot(\vec{b}\times\vec{d})=0$.
33. The acute angle θ between the two planes $\vec{r}\cdot\vec{n}_1=p_1$ and $\vec{r}\cdot\vec{n}_2=p_2$ is $\theta=\cos^{-1}\left(\dfrac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}\right)$

34. If θ is the acute angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = p$, then $\theta = \sin^{-1}\left(\frac{\left|\vec{b} \cdot \vec{n}\right|}{\left|\vec{b}\right|\left|\vec{n}\right|}\right)$

35. The perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r} \cdot \vec{n} = p$ is given by $\delta = \frac{|\vec{u} \cdot \vec{n} - p|}{|\vec{n}|}$

36. The perpendicular distance from a point (x_2, y_1, z_1) to the plane $ax + by + cz = p$ is $\delta = \frac{|ax_1 + by_1 + cz_1 - p|}{\sqrt{a^2 + b^2 + c^2}}$.

37. The perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is given by

$$\delta = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

38. The distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

39. The vector equation of a plane which passes through the line of intersection of the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by $(\vec{r} \cdot \vec{n}_1 - d_1) + \lambda(\vec{r} \cdot \vec{n}_2 - d_2) = 0$, where $\lambda \in \mathbb{R}$ is an.

40. The equation of a plane passing through the line of intersection of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by

$$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$$

41. The position vector of the point of intersection of the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = p$ is $\vec{u} = \vec{a} + \left(\frac{p - (\vec{a} \cdot \vec{n})}{\vec{b} \cdot \vec{n}}\right)\vec{b}$, where $\vec{b} \cdot \vec{n} \neq \vec{0}$.

42. If $\vec{v}$ is the position vector of the image of $\vec{u}$ in the plane $\vec{r} \cdot \vec{n} = p$, then $\vec{v} = \vec{u} + \frac{2\left[p - (\vec{u} \cdot \vec{n})\right]}{|n|^2}\vec{n}$.

ICT CORNER

https://ggbm.at/vchq92pg or Scan the QR Code

Open the Browser, type the URL Link given below (or) Scan the QR code. GeoGebra work book named "12th Standard Mathematics" will open. In the left side of the work book there are many chapters related to your text book. Click on the chapter named "Applications of Vector Algebra". You can see several work sheets related to the chapter. Select the work sheet "Scalar Triple Product"

12th_Maths_EM_Vol1_CH 6 Vector Algebra.indd 281 07-12-2021 13:20:02

ANSWERS

Exercise 1.1

1. (i) $\begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$ (iii) $\frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & +2 & 2 \end{bmatrix}$

2. (i) $\frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$ (ii) $\frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$ (iii) $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$ 4. $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

8. $\pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$ 9. $\pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ 10. $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$ 13. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 15. HELP

Exercise 1.2

1. (i) 1 (ii) 2 (iii) 2 (iv) 3 (v) 3 2. (i) 2 (ii) 3 (iii) 3

3. (i) $\begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Exercise 1.3

1. (i) $x = -11, y = 4$ (ii) $x = 2, y = -4$
   (iii) $x = 2, y = 3, z = 4$ (iv) $x = 3, y = -2, z = 1$
2. $x = 2, y = 1, z = -1$ 3. ₹ 18000, ₹ 600
4. 18 days, 36 days 5. ₹ 2000, ₹ 1000, ₹ 3000

Exercise 1.4

1. (i) $x = -2, y = 3$ (ii) $x = \frac{1}{2}, y = 3$
   (iii) $x = 2, y = 3, z = 4$ (iv) $x = 1, y = 3, z = 3$
2. 84 3. 50% acid is 6 litres, 25% acid is 4 litres
4. Pump A : 15 minutes, Pump B : 30 minutes
5. ₹ 30/-, ₹ 10/-, ₹ 30/-, yes

Exercise 1.5

1. (i) $x = -1, y = 4, z = 4$ (ii) $x = 3, y = 1, z = 2$
2. $a = 2, b = 1, c = 6$ 3. ₹ 30000, ₹ 15000, ₹ 20000
4. $a = 1, b = 3, c = -10$, yes

Exercise 1.6

1. (i) $x = y = z = 1$ (ii) $x = \frac{1}{10}(7 - 5t), y = \frac{1}{10}(5t - 1), z = t, t \in \mathbb{R}$

(iii) No solution (iv) $x=\frac{1}{2}(s-t+2), y=s, z=t$ and $s,t\in\mathbb{R}$

2. (i) $k=1$ (ii) $k\neq 1, k\neq -2$ (iii) $k=-2$
3. (i) $\lambda = 5$ and $\mu \neq 9$ (ii) $\lambda \neq 5$ and $\mu \in \mathbb{R}$ (iii) $\lambda=5, \mu=9$

Exercise 1.7

1. (i) $x=-t, y=-2t, z=t, t\in\mathbb{R}$ (ii) Trivial solutions only
2. (i) $\lambda\neq 8$ (ii) $\lambda=8$
3. $2C_2H_6+7O_2 \rightarrow 6H_2O+4CO_2$

Exercise 1.8

1	2	3	4	5	6	7	8	9	10
(2)	(3)	(2)	(3)	(4)	(2)	(4)	(4)	(2)	(1)
11	12	13	14	15	16	17	18	19	20
(2)	(4)	(1)	(2)	(4)	(3)	(2)	(1)	(4)	(4)
21	22	23	24	25					
(2)	(4)	(4)	(4)	(1)					

Exercise 2.1

1. $-1-i$ 2. $1+i$ 3. 0 4. 0
5. 1 6. $1-i$

Exercise 2.2

1. (i) $4+i$ (ii) $8-i$ (iii) $7+5i$ (iv) $1+17i$
(v) $15+8i$ (vi) $15+8i$
3. $x=-1$, $y=1$

Exercise 2.3

3. $-z_1=-2-5i$, $z_1^{-1}=\frac{1}{29}(2-5i)$

$-z_2=3+4i$, $z_2^{-1}=\frac{1}{25}(-3+4i)$

$-z_3=-1-i$, $z_3^{-1}=\frac{1}{2}(1-i)$

Exercise 2.4

1. (i) $7-5i$ (ii) $\frac{5}{4}(1-i)$ (iii) $\frac{2}{5}-\frac{14i}{5}$
2. (i) $\frac{x}{x^2+y^2}$ (ii) y (iii) $-y-4$
3. $\frac{1}{25}(-1-2i)$, $\frac{1}{5}(-11+2i)$ (4) $\frac{1}{2}(7-i)$
6. (i) 6 (ii) 3

Exercise 2.5

1. (i) $\frac{2}{5}$ (ii) $2\sqrt{2}$ (iii) 32 (iv) 50
3. $11+6i$
8. 10 10. (i) $\pm\left(\frac{3}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$ (ii) $\pm\left(\sqrt{2}+i2\sqrt{2}\right)$ (iii) $\pm(2-3i)$

Exercise 2.6

3. (i) $y^2=3$ (ii) $x-y=0$ (iii) $x+y=0$ (iv) $x^2+y^2=1$
4. (i) $2+i, 3$ (ii) $-1+2i, 1$ (iii) $2-4i, \frac{8}{3}$
5. (i) $x^2+y^2-8x-240=0$ (ii) $6x+1=0$

Exercise 2.7

1. (i) $4\left(\cos\left(2k\pi+\frac{\pi}{3}\right)+i\sin\left(2k\pi+\frac{\pi}{3}\right)\right), k\in\mathbb{Z}$

   (ii) $2\sqrt{3}\left(\cos\left(2k\pi-\frac{\pi}{6}\right)+i\sin\left(2k\pi-\frac{\pi}{6}\right)\right), k\in\mathbb{Z}$

   (iii) $2\sqrt{2}\left(\cos\left(2k\pi-\frac{3\pi}{4}\right)+i\sin\left(2k\pi-\frac{3\pi}{4}\right)\right), k\in\mathbb{Z}$

   (iv) $\sqrt{2}\left(\cos\left(2k\pi+\frac{5\pi}{12}\right)+i\sin\left(2k\pi+\frac{5\pi}{12}\right)\right), k\in\mathbb{Z}$

2. (i) $\frac{1}{\sqrt{2}}(1+i)$ (ii) $\frac{-i}{2}$

Exercise 2.8

3. 1 5. $3\,\text{cis}\frac{\pi}{3}, -3, 3\,\text{cis}\frac{5\pi}{3}$ 7. -1

9. (i) $2\sqrt{2}\,e^{\frac{i\pi}{12}}$ (ii) $2\sqrt{2}\,e^{\frac{i5\pi}{12}}$ (iii) $2\sqrt{2}\,e^{\frac{i5\pi}{4}}$

Exercise 2.9

1	2	3	4	5	6	7	8	9	10
(1)	(1)	(1)	(2)	(3)	(1)	(4)	(1)	(1)	(1)
11	12	13	14	15	16	17	18	19	20
(2)	(2)	(4)	(2)	(2)	(3)	(1)	(3)	(4)	(4)
21	22	23	24	25					
(2)	(3)	(4)	(1)	(1)					

EXERCISE 3.1

1. 60
2. (i) $x^3-6x^2+11x-6=0$ (ii) $x^3-3x+2=0$ (iii) $x^3-4x^2-4x+16=0$
3. (i) $x^3+4x^2+12x+32=0$ (ii) $4x^3+3x^2+2x+1=0$ (ii) $x^3-2x^2+3x-4=0$

4. $2,3,\frac{1}{3}$ 5. 10 6. $6,4,-1$ 7. $\sum\frac{\alpha}{\beta\gamma}=\frac{2ac-b^2}{ad}$

8. $2x^2-3x-20=0$ 11. $x^3+x-12=0$

Exercise 3.2

1. When $k<0$, the polynomial has real roots.
   When $k=0$ or $k=8$, the roots are real and equal.
   When $0<k<8$ the roots are imaginary.
   When $k>8$ the roots are real and distinct.
2. $x^2-4x+7=0$ 3. $x^2-6x+13=0$ 4. x^4-16x^2+4

Exercise 3.3

1. $-3,3,\frac{1}{2}$ 2. $\frac{2}{3},\frac{4}{3},2$ 3. $\frac{2}{3},2,6$ 4. $k=2,\ 2,\frac{1\pm\sqrt{3}}{2}$

5. $1-2i,1+2i,\sqrt{3},-\sqrt{3},\frac{1+\sqrt{37}}{2},\frac{1-\sqrt{37}}{2}$ 6. (i) $1,\frac{1}{2},3$ (ii) $-1,\frac{1}{2},\frac{3}{4}$ 7. $\pm3,\pm\sqrt{5}$

Exercise 3.4

1. (i) $\{-2,3,-7,8\}$ (ii) $3,3,3+\sqrt{17},3-\sqrt{17}$ 2. $\left\{1,-2,\frac{-1+2\sqrt{5}}{2},\frac{-1-2\sqrt{5}}{2}\right\}$

Exercise 3.5

1. (i) $n\pi+(-1)^n\frac{\pi}{2},n\in\mathbb{Z}$, no solution for $\sin x=4$ (ii) $2,-\frac{1}{4},\frac{2}{3}$

2. (i) $x=1$ (ii) no rational roots 3. 4^n 4. $\frac{b^2}{4a},\frac{9a^3}{b^2}$

5. (i) $2,3,\frac{1}{2},\frac{1}{3}$ (ii) $+1,-1,\frac{-3+\sqrt{5}}{2},\frac{-3-\sqrt{5}}{2}$

6. $2,3$ 7. $\frac{1}{3},3,-\frac{1}{2}$ and -2

Exercise 3.6

1. It has atmost four positive roots and atmost three negative roots.
2. It has atmost two positive roots and no negative roots.
4. It has one positive real root and one negative real root.
5. no positive real roots and no negative real roots.

Exercise 3.7

1	2	3	4	5	6	7	8	9	10
(4)	(1)	(3)	(1)	(3)	(4)	(1)	(3)	(1)	(2)

Exercise 4.1

1. (i) $x = n\pi,\ n = 0, \pm1, \pm2, \ldots \pm10$ (ii) $x = (4n-1)\dfrac{\pi}{2},\ n = 0,\ \pm1$

2. (i) $1,\ \dfrac{2\pi}{7}$ (ii) $1,\ 6\pi$ (iii) $4,\ \pi$ 4. (i) $\dfrac{\pi}{3}$ (ii) $-\dfrac{\pi}{4}$

5. $x = 0$ 6. (i) $\{-1,\ 1\}$ (ii) $[0,\ 1]$ 7. $\dfrac{\pi}{3}$

Exercise 4.2

1. (i) $x = (2n+1)\dfrac{\pi}{2},\ n = 0, \pm1, \pm2, \pm3, \pm4, \pm5, -6$ (ii) $x = (2n+1)\pi,\ n = 0, \pm1, \pm2, -3$

2. $-\dfrac{\pi}{6} \notin [0,\ \pi]$ 3. True 4. $\dfrac{\pi}{3}$

5. (i) $\dfrac{5\pi}{6}$ (ii) $-\dfrac{\pi}{6}$ (iii) $\dfrac{24\pi}{119}$ 6. (i) $[-5,\ 5]$ (ii) $[-1,\ 1]$

7. $0 < x < \dfrac{1}{3}$ 8. (i) 0 (ii) $\dfrac{17\pi}{12}$

Exercise 4.3

1. (i) $[-3,\ 3]$ (ii) $\mathbb{R}$ 2. (i) $\dfrac{\pi}{4}$ (ii) $-\dfrac{\pi}{6}$

3. (i) $\dfrac{7\pi}{4}$ (ii) 1947 (iii) -0.2021 4. (i) ∞ (ii) $-\dfrac{2\sqrt{5}}{25}$ (iii) $\dfrac{24}{25}$

Exercise 4.4

1. (i) $\dfrac{\pi}{6}$ (ii) $\dfrac{\pi}{6}$ (iii) $-\dfrac{\pi}{4}$ 2. (i) $-\dfrac{\pi}{3}$ (ii) $\cot^{-1}(2) - \dfrac{\pi}{6}$ (iii) $-\dfrac{5\pi}{6}$

Exercise 4.5

1. (i) $-\dfrac{\pi}{2}$ (ii) $-\dfrac{\pi}{4}$ (iii) $5 - 2\pi$

2. (i) $\sqrt{2x - x^2}$ (ii) $\dfrac{1}{\sqrt{9x^2 - 6x + 2}}$ (iii) $\dfrac{2x+1}{\sqrt{3 - 4x - 4x^2}}$

3. (i) $\dfrac{\pi}{6}$ (ii) 0 (iii) $\dfrac{17}{6}$ 8. $\dfrac{\pi}{4}$

9. (i) $x = 13$ (ii) $x = \dfrac{a-b}{1+ab}$ (iii) $x = 2n\pi,\ x = n\pi + \dfrac{\pi}{4},\ n \in \mathbb{Z}$

(iv) $x = \sqrt{3}$

10. 3

Exercise 4.6

1	2	3	4	5	6	7	8	9	10
(3)	(2)	(3)	(1)	(2)	(1)	(3)	(1)	(4)	(4)
11	12	13	14	15	16	17	18	19	20
(3)	(2)	(2)	(1)	(3)	(3)	(2)	(2)	(4)	(4)

Exercise 5.1

1. $x^2+y^2 \pm 10y=0$ 2. $(x-2)^2+(y+1)^2=50$

3. $x^2+y^2+4x+4y+4=0$ or $x^2+y^2+20x+20y+100=0$ 4. $x^2+y^2-4x-6y-12=0$

5. $x^2+y^2-5x+3y-22=0$ 6. $x^2+y^2=1$

7. $x^2+y^2-6x-4y+4=0$ 8. ± 12

9. $x-5y+8=0,\ 5x+y-12=0$ 10. out side, inside, outside

11. (i) $(0,-2),0$ (ii) $(-3,2),3$ (iii) $\left(\frac{1}{2},-1\right),\frac{\sqrt{17}}{2}$ (iv) $\left(\frac{3}{2},-1\right),\frac{3}{2}$

12. $p=q=3,\ (1,0),5$

Exercise 5.2

1. (i) $y^2=16x$ (ii) $3x^2=-4y$ (iii) $(y+2)^2=12(x-1)$ (iv) $y^2=16x$

2. (i) $\frac{x^2}{36}+\frac{y^2}{27}=1$ (ii) $\frac{x^2}{9}+\frac{y^2}{25}=1$ (iii) $\frac{16x^2}{625}+\frac{y^2}{25}=1$ (iii) $\frac{x^2}{8}+\frac{y^2}{16}=1$

3. (i) $\frac{9x^2}{16}-\frac{9y^2}{20}=1$ (ii) $\frac{(x-2)^2}{12}-\frac{(y-1)^2}{24}=1$ (iii) $\frac{x^2}{16}-\frac{9y^2}{64}=1$

4.

	Vertex	Focus	Equation of directrix	Length of latus rectum
i.	$(0,0)$	$(4,0)$	$x=-4$	16
ii.	$(0,0)$	$(0,6)$	$y=-6$	24
iii.	$(0,0)$	$(-2,0)$	$x=2$	8
iv.	$(1,-2)$	$(1,-4)$	$y=0$	8
v.	$(1,2)$	$(3,2)$	$x=-1$	8

5.

	Type of conic	Centre	Vertices	Foci	Directrices
i.	Ellipse	$(0,0)$	$(\pm 5,0)$	$(\pm 4,0)$	$x=\pm\frac{25}{4}$
ii.	Ellipse	$(0,0)$	$(0,\pm\sqrt{10})$	$(0,\pm\sqrt{7})$	$y=\pm\frac{10}{\sqrt{7}}$
iii.	Hyperbola	$(0,0)$	$(\pm 5,0)$	$(\pm 13,0)$	$x=\pm\frac{25}{13}$
iv.	Hyperbola	$(0,0)$	$(0,\pm 4)$	$(0,\pm 5)$	$y=\pm\frac{16}{5}$

8.

	Type of Conic	Centre	Vertices	Foci	Directrices
i.	Ellipse	$(3,4)$	$(3,21),(3,-13)$	$(3,12),(3,-4)$	$y=\frac{321}{8}$, $y=\frac{-257}{8}$
ii.	Ellipse	$(-1,2)$	$(-11,2),(9,2)$	$(-7,2),(5,2)$	$x=\frac{47}{3}$, $x=\frac{-53}{3}$
iii.	Hyperbola	$(-3,4)$	$(-18,4),(12,4)$	$(-20,4),(14,4)$	$x=\frac{174}{17}$, $x=\frac{-276}{17}$
iv.	Hyperbola	$(-1,2)$	$(-1,7),(-1,-3)$	$(-1,2+\sqrt{41})$ $(-1,2-\sqrt{41})$	$y=\frac{25}{\sqrt{41}}+2$, $y=\frac{-25}{\sqrt{41}}+2$
v.	Ellipse	$(4,-2)$	$(4,-2+3\sqrt{2})$, $(4,-2-3\sqrt{2})$	$(4,-2+\sqrt{6})$, $(4,-2-\sqrt{6})$	$y=-2+3\sqrt{6}$, $y=-2-3\sqrt{6}$
vi.	Hyperbola	$(2,-3)$	$(3,-3),(1,-3)$	$(2+\sqrt{10}-3)$, $(2-\sqrt{10},-3)$	$x=\frac{1}{\sqrt{10}}+2$, $x=\frac{-1}{\sqrt{10}}+2$

Exercise 5.3

1. hyperbola 2 circle 3. ellipse 4. circle 5. hyperbola 6. parabola

Exercise 5.4

1. $x-y-3=0, x-9y+13=0$
2. $10x-3y+32=0,\ 10x-3y-32=0$
3. $(-3,1)$
4. $x-y+4=0$
5. $x-2y+8=0$
6. $4x-3y-6=0, 3x+4y-42=0$

Exercise 5.5

1. 8.4 m
2. 26.6 m
3. 3 m
4. $y^2=4.8x,\ 1.3m$
5. $3.52m,\ 5.08m$
6. 90.82m, 148.91m
7. $\frac{2\sqrt{2}}{3}$
8. $3\sqrt{3}m$
9. $\tan^{-1}\left(\frac{4}{3}\right)$
10. $\frac{x^2}{9}-\frac{y^2}{16}=1$

Exercise 5.6

1	2	3	4	5	6	7	8	9	10	11	12	13
(1)	(3)	(4)	(3)	(3)	(1)	(1)	(3)	(2)	(2)	(1)	(4)	(3)
14	15	16	17	18	19	20	21	22	23	24	25	---
(3)	(1)	(4)	(4)	(1)	(1)	(2)	(2)	(3)	(3)	(3)	(2)	---

Exercise 6.1

11. 80 units
12. 69 units
13. $\sqrt{179},\ \frac{3}{\sqrt{179}},\ \frac{-11}{\sqrt{179}},\ \frac{-7}{\sqrt{179}}$
14. $-96\hat{i}+115\hat{j}+15\hat{k}$

Exercise 6.2

1. 24
2. 720 cubic units
3. -5
4. ± 12
5. $\frac{2\sqrt{3}}{5}$
6. coplanar
7. 2

Exercise 6.3

1. (i) $-2\hat{i}+14\hat{j}-22\hat{k}$ (ii) $22\hat{i}+14\hat{j}+2\hat{k}$
5. -74
7. $l=0,\ m=10,\ n=-3$
8. $\theta=\frac{\pi}{3}$

Exercise 6.4

1. $\left(\vec{r}-\left(4\hat{i}+3\hat{j}-7\hat{k}\right)\right)\times\left(2\hat{i}-6\hat{j}+7\hat{k}\right)=\vec{0}$, $\frac{x-4}{2}=\frac{y-3}{-6}=\frac{z+7}{7}$
2. $\vec{r}=\left(-2\hat{i}+3\hat{j}+4\hat{k}\right)+t\left(-4\hat{i}+5\hat{j}-6\hat{k}\right)$, $\frac{x+2}{-4}=\frac{y-3}{5}=\frac{z-4}{-6}$
3. $\left(\frac{32}{3}, 0, \frac{47}{3}\right)$, $(0,16,-11)$

4. $\left(\frac{2}{7},\frac{3}{7},\frac{6}{7}\right)$, $\vec{r}=\left(5\hat{i}+6\hat{j}+7\hat{k}\right)+t\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$ or

$\vec{r}=\left(7\hat{i}+9\hat{j}+13\hat{k}\right)+t\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$,

$\frac{x-5}{2}=\frac{y-6}{3}=\frac{z-7}{6}$ or $\frac{x-7}{2}=\frac{y-9}{3}=\frac{z-13}{6}$

5. (i) 0° (ii) $\frac{\pi}{6}$ (iii) $\frac{\pi}{2}$ 6. $\frac{\pi}{2}$ 7. $a=18, b=\frac{2}{3}$ 8. 1

Exercise 6.5

1. $\vec{r}=\left(5\hat{i}+2\hat{j}+8\hat{k}\right)+t\left(2\hat{i}+\hat{j}-2\hat{k}\right), t\in\mathbb{R}$, $\frac{x-5}{2}=\frac{y-2}{1}=\frac{z-8}{-2}$

2. $\frac{7}{\sqrt{5}}$ units 3. $\frac{9}{2}$ 4. $(6,2,1)$ 5. 2 units

6. $\frac{\sqrt{83}}{\sqrt{6}}$ units 7. $(1,6,0)$, $\frac{x-5}{-4}=\frac{y-4}{2}=\frac{z-2}{-2}$

Exercise 6.6

1. $\vec{r}\cdot\left(\frac{3\hat{i}-4\hat{j}+5\hat{k}}{5\sqrt{2}}\right)=7$ 2. $\frac{12}{13},\frac{3}{13},\frac{-4}{13}$; $\vec{r}.\left(\frac{12\hat{i}+3\hat{j}-4\hat{k}}{13}\right)=5$; 5

3. $\vec{r}\cdot\left(\hat{i}+3\hat{j}+5\hat{k}\right)=35$; $x+3y+5z=35$ 4. $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=2$; $x+y+z=2$

5. x-intercept $=2$, y- intercept $=3$, z- intercept $=-4$ 6. $\frac{x}{u}+\frac{y}{v}+\frac{z}{w}=3$

Exercise 6.7

1. $\vec{r}\cdot\left(\hat{i}-2\hat{j}+4\hat{k}\right)=20$; $x-2y+4z-20=0$

2. $\vec{r}\cdot\left(3\hat{i}+4\hat{j}-5\hat{k}\right)=9$; $3x+4y-5z-9=0$

3. $\vec{r}=\left(2\hat{i}+2\hat{j}+\hat{k}\right)+s\left(-\hat{i}-4\hat{j}+2\hat{k}\right)+t\left(3\hat{i}-4\hat{j}+5\hat{k}\right) s,t\in\mathbb{R}$;

$12x-11y-16z+14=0$

4. $\vec{r}\cdot\left(\hat{i}+10\hat{j}+7\hat{k}\right)=9$; $x+10y+7z-9=0$

5. $\vec{r}=\left(\hat{i}-\hat{j}+3\hat{k}\right)+s\left(2\hat{i}-\hat{j}+4\hat{k}\right)+t\left(\hat{i}+2\hat{j}+\hat{k}\right) s,t\in\mathbb{R}$; $9x-2y-5z+4=0$

6. $\vec{r}=\left(3\hat{i}+6\hat{j}-2\hat{k}\right)+s\left(-4\hat{i}-8\hat{j}+8\hat{k}\right)+t\left(3\hat{i}-2\hat{j}\right) s,t\in\mathbb{R}$; $\vec{r}\cdot\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=16$;

$2x+3y+4z-16=0$

7. $\vec{r}\cdot\left(3\hat{i}+5\hat{j}-7\hat{k}\right)=6$; $3x+5y-7z-6=0$

Exercise 6.8

1. $\vec{r}\cdot\left(17\hat{i}-47\hat{j}-24\hat{k}\right)=-172$ 2. $x+2y-z-4=0$

3. $m=\pm\sqrt{2}$ 4. $-2, 2$, $y+z+1=0$, $y-z+1=0$

Exercise 6.9

1. $15x-47y+28z-7=0$
2. $5x-11y+z-17=0$
3. $\sin^{-1}\left(\frac{8}{21}\right)$
4. $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$
5. $2x-3y+5z+11=0, \frac{4}{\sqrt{38}}$
6. $\frac{1}{\sqrt{3}}$ units
7. $(2,2,0)$, $\text{Sin}^{-1}\left(\frac{2}{3\sqrt{6}}\right)$
8. $(3,1,-1)$; $\sqrt{14}$ units.

Exercise 6.10

1	2	3	4	5	6	7	8	9	10
(4)	(3)	(1)	(2)	(1)	(3)	(1)	(1)	(1)	(2)
11	12	13	14	15	16	17	18	19	20
(3)	(1)	(2)	(4)	(4)	(2)	(3)	(4)	(2)	(1)
21	22	23	24	25					
(2)	(3)	(4)	(3)	(1)					

GLOSSARY

CHAPTER 1

APPLICATIONS OF MATRICES AND DETERMINANTS

Adjoint matrix	சேர்ப்பு அணி
Inverse matrix	நேர்மாறு அணி
Rank	தரம்
Elementary transformation	சாதாரண உருமாற்றங்கள்
Echelon form	ஏறுபடி வடிவம்
Trivial solution	வெளிப்படைத் தீர்வு
Non-trivial solution	வெளிப்படையற்ற தீர்வு
Augmented matrix	விரிவுபடுத்தப்பட்ட அணி
Consistent	ஒருங்கமைவு உடையது
Pivot	சுழற்சித்தான உறுப்பு

CHAPTER 2

COMPLEX NUMBER

Complex numbers	கலப்பு எண்கள்
Imaginary unit	கற்பனை அலகு
Rectangular form	செவ்வக வடிவம்
Argand Plane	ஆர்கன்ட் தளம்
Conjugate of a complex number	ஒரு கலப்பெண்ணின் இணைக் கலப்பெண்
Upper bound	மேல் எல்லை
Lower bound	கீழ் எல்லை
Polar form	துருவ வடிவம்
Exponential form	அடுக்குக்குறி வடிவம்
Trigonometric form	முக்கோணவியல் வடிவம்
Absolute value	எண்ணளவு
Modulus	மட்டு மதிப்பு
Argument or amplitude	வீச்சு
Principal argument	முதன்மை வீச்சு
Euler's form	ஆய்லர் வடிவம்

CHAPTER 3

THEORY OF EQUATION

Complex conjugate root theorem	இணைக்கலப்பெண் மூலத் தேற்றம்
Leading coefficient	முதன்மைக் கெழு
Leading term	முதன்மை உறுப்பு
Monic polynomial	ஒற்றை முதன்மை உறுப்பு பல்லுறுப்புக் கோவை
Non-polynomial equation	பல்லுறுப்புக் கோவையற்ற சமன்பாடு

Non real complex number	மெய்யற்ற கலப்பெண்
Quartic polynomial	நாற்படி பல்லுறுப்புக் கோவை
Radical solution	அடிப்படைத்தீர்வு
Rational root Theorem	விகிதமுறு மூலத்தேற்றம்
Reciprocal equation	தலைகீழ் சமன்பாடு
Reciprocal polynomial	தலைகீழ் பல்லுறுப்புக் கோவை
Simple root	எளிய மூலம்
Zero polynomial	பூஜ்ஜிய பல்லுறுப்புக் கோவை

CHAPTER 4

INVERSE TRIGONOMETRIC FUNCTION

Inverse trigonometric functions	நேர்மாறு முக்கோணவியல் சார்புகள்
Principal value	முதன்மை மதிப்பு
Amplitude	வீச்சு
Period	காலம்
Principal domain	முதன்மை சார்பகம்
Periodic function	காலமுறைச் சார்பு
Reciprocal inverse identities	நேர்மாறு தலைகீழி முற்றொருமைகள்
Reflection identities	பிரதிபலிப்பு முற்றொருமைகள்
Cofunction inverse identities	நேர்மாறு துணைச் சார்பு முற்றொருமைகள்

CHAPTER 5

ANALYTICAL GEOMETRY II AND FUNCTIONS

Circle	வட்டம்
Parabola	பரவளையம்
Ellipse	நீள்வட்டம்
Hyperbola	அதிபரவளையம்
Algebraic techniques	இயற்கணித நுட்பங்கள்
Geometrical problems	வடிவியல் கணக்குகள்
Astronomy	வானியல்
Conics	கூம்பு வளைவுகள்
Focus	குவியம்
Directrix	இயக்குவரை
Eccentricity	மையத்தொலைத் தகவு

Focal chord	குவி நாண்
Vertex	முனை
Latus rectum	செவ்வகலம்
Major axis	நெட்டச்சு
Minor axis	குற்றச்சு
Transverse axis	துணையச்சு
Conjugate axis	குறுக்கச்சு
Auxiliary circle	துணை வட்டம்
Incircle	உள் வட்டம்
Asymptotes	தொலைத் தொடுகோடுகள்
Degenerate forms	சிதைந்த வடிவங்கள்
Double napped cone	இரட்டைக் கூம்பு
Parametric equation	துணையலகுச் சமன்பாடுகள்
Director circle	இயக்கு வட்டம்
Elliptic orbit	நீள்வட்ட சுற்றுப்பாதை
Reflective property	பிரதிபலிப்பு பண்புகள்

CHAPTER 6

VECTOR ALGEBRA AND ITS APPLICATIONS

Box product	பெட்டிப் பெருக்கல்
Line of intersection	வெட்டுக்கோடு
Moment	திருப்புத்திறன்
Normal	செங்குத்து
Parallelepiped	இணைகரத்திண்மம்
Parameter	துணையலகு
Plane	தளம்
Rotaional force	சுழல் விசை
Skew lines	ஒரு தள அமையாக் கோடுகள்
Torque	திருப்பு விசை
Triple product	முப்பெருக்கல்

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